EXISTENCE RESULTS FOR IMPULSIVE NEUTRAL PARTIAL DIFFERENTIAL INCLUSIONS WITH STATE DEPENDENT DELAY IN BANACH SPACE

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Abstract: In this paper, we prove the existence of mild solutions for a first order impulsive neutral differential inclusion with state dependent delay. By using a fixed point theorem for condensing multi-valued maps, a main existence theorem is established.

Keywords- Impulsive partial neutral differential inclusions, state-dependent delay; fixed point.
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I.INTRODUCTION

The theory of impulsive differential equations appears as a neutral description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade, see the monographs of Benchohra et al. [4], Haddad et al.[10], Lakshmikantham et al. [16], the papers [1,3, 5, 6, 7,8, 15,19,21] and the references therein.

Neutral differential systems with impulses arise in many areas of applied mathematics and for this reason these systems have been extensively investigated in the last decades. Recently, much attention has been paid to existence results for partial functional differential equations with state-dependent delay, and cite the works [2, 11, 12, 13, 14, 17, 18, 20] and the references therein. To the best of our knowledge, few papers can be found in the literature on the existence of mild solutions for an abstract impulsive differential inclusion with State delay. In the present paper we consider existence results for impulsive neutral differential inclusions with state-dependent delay such as

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$$\frac{d}{dt}[x(t)-g(t,x_t)] \in Ax(t)+F[t,x_{\rho(t,x_t)}],\ t \neq t_i,\ t \in J=[0,a] \tag{1.1}$$

$$x_0=\phi \in B \tag{1.2}$$

$$\Delta x(t_i)=I_i(x_{t_i}),\ t=t_i\ i=1,2,....,n \tag{1.3}$$

where A is the infinitesimal generator of a compact, analytic semigroup $T(t),\ t>0$ in a Banach space $X; F: J\times B \to P(X)$ is a bounded closed convex-valued multi-valued map, $P(X)$ is the family of all nonempty subsets of $X; g: J\times B \to X$, $\rho: J\times B \to (-\infty, a]$, $I_i: B \to X$, $i=1,2,...,n$, are appropriate functions, where B is an abstract phase space defined below, $0=t_0<t_1<...<t_n<t_{n+1}=a$, $\Delta\xi(t)=\xi(t^+)-\xi(t^-)$, $\xi(t^-)$ and $\xi(t^+)$ represent the left and right limits of $\xi(t)$ at t. The histories

$$x_t:(-\infty,0]\to X,\ x_t(s)=x(t+s),\ s\le 0,$$

belong to the abstract phase space B.

II. PRELIMINARIES

In this section, we introduce some basic definitions, notations and results which are used throughout this paper.

Let $C(J,K)$ be the Banach space of continuous functions y from J into X with the norm $\|y\|_\infty=\sup\{\|y(t)\|: t\in J\}$. $L(X)$ denotes the Banach space of bounded linear operators from X into itself. A measurable function $y: J\to X$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. For properties of the Bochner integral see Yosida [22]. $L^1(J,X)$ is the Banach space of continuous functions $y: J \to X$ which are Bochner integrable and equipped with the norm $\|y\|_{L^1}=\int_0^a \|y(t)\|dt.$

A multi-valued map $G: X\to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. G is bounded on bounded sets if $G(B)=\cup_{X\in B}\ G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\|: y\in G(x)\}\}<\infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set Ω of X containing $G(x_0)$, there exists an open neighbourhood V of x_0 such that $G(V)\subseteq\Omega$.

G is said to be completely continuous if $G(\Omega)$ is relatively compact for every bounded subset Ω of X.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n\to x_*,\ y_n\to y_*, y_n\in G(x_n)$ imply $y_*\in G(x_*)$).

An upper semi-continuous multi-valued map $G: X\to P(X)$ is said to be condensing [9] if for any subset $B \subset X$ with $N(B) \neq 0$ we have $N(G(B)) < N(B)$, where N denotes the Kuratowski measure of non-compactness [3].

Let $P_{cp,cv}(X)$ denote the classes of all bounded and compact convex subsets of X.

G has a fixed point if there is an $x \in X$ Such that $x \in G(X)$. For more details on multi-valued maps, see the books of Deimling [9] and Hu and Papageorgiou [15].

Let $\wp C$ forbmed by all functions $y:[0,a] \to X$ such that y is continuous at $t \neq t_k, y(t_k^-) = y(t_k)$ and $y(t_k^+)$ exists for all $k = 1,2,...,n$. In this paper we always assume that $\wp C$ is endowed with the norm $\|y\|_{\wp C} = \sup_{s \in J} \|y(s)\|$. It is clear that $(\wp C, \|.\|_{\wp C})$ is a Banach space.

To set the framework for our main existence results, we need to introduce the following definitions and lemmas. In this work we will employ an axiomatic definition for the phase spase B which is introduced in [11]. Specifically, B will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|.\|_B$, and satisfies the following axioms:

(A) If $x:(-\infty, \sigma + a) \to X$, $a > 0$, is such that $x_{[\sigma,\sigma+a]} \in \wp C([\sigma, \sigma + a], X)$ and $x_\sigma \in B$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold:

(i) x_t *is in* B.

(ii) $\|x(t)\| \leq H \|x_t\|_B$.

(iii) $\|x_t\|_B \leq K(t-\sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_B$, where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K continuous. M is locally bounded, and H, K, M are independent of $x(.)$.

(B) The space B is complete.

Definition 2.1. A function $x:(-\infty, a) \to X$, is called a mild solution of the problem (1.1)-(1.3) if $x_0 = \phi$, $x_{\rho(s,x_s)} \in B$ for every $s \in J$ and $\Delta x(t_i) = I_i(x_{t_i})$, $i = 1,2,...n$ the function $AT(t-s)g(s,x_s)$ is Bochner integrable and the impulsive integral inclusion

$$x(t) \in T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds$$

$$+ \int_0^t T(t-s)F(s, x_{\rho(s,x_s)})ds + \sum_{0<t_i<t} T(t-t_i)I_i(x_{t_i})$$

is satisfied.

Lemma 2.1. [17] Let X be a Banach space. Let $F : J \times B \to \rho_{cp,cv}(X)$ satisfy

(i) The function $F(.,\psi) : J \to X$ is measurable for every $\psi \in B$

(ii) The function $F(t,.) : B \to X$ is u.s.c. for each $t \in J$.

(iii) For each fixed $\psi \in B$, the set
$S_{F,\psi} = \{ f \in L^1(J,X) : f(t) \in F(t,\psi) \text{ for a.e } t \in J \}$
is nonempty.

Let Γ be a linear continuous mapping from $L^1(J,X) \to C(J,X)$. Then the operator $\Gamma o S_F : C(J,X) \to \rho_{cp,cv}(C(J,X))$ $y \to (\Gamma o S_F)(y) : \Gamma(S_F, y)$ is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2.2. [9] Let B be bounded and convex set in Banach space X. $\Gamma : B \to P(B)$ is a u.s.c., condensing multi-valued map. If for every $x \in B$, $\Gamma(x)$ is closed and convex set in B, then Γ has a fixed point in B.

III. EXISTENCE RESULTS

Throughout this section, $A : D(A) \to X$ will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)(t>0)$. Let $0 \in \rho(A)$, Then it is possible to define the fractional power A^α, for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Hereafter we denote by X_α the Banach space $D(A^\alpha)$ normed with $\|x\|_\alpha$. Then for each $0 < \alpha \le 1$, X_α is Banach space, and $X_\alpha \to X_\beta$ for $0 < \beta < \alpha \le 1$, and the imbedding is compact whenever the resolvent operator of A is compact. For semigroup $\{T(t)_{t \ge 0}\}$, the following properties will be used:

(a) there is a $M \ge 1$ such that $\|T(t)\| \le M$ for all $0 \le t \le a$;

(b) for any $0 \le a \le t$, there exists a positive constant C_α such that

$$\|A^\alpha T(t)\| \le \frac{C_\alpha}{t^\alpha}, \quad 0 < t \le a.$$

For more details about the above preliminaries, we refer to [22].

In order to define the solution of system $(1.1)-(1.3)$ we shall consider the space $\Omega = \{x, [0,a] \to X_\alpha : x_k \in C(J_k, X_{\alpha,}),\ k = 0,1,...m,$ and there exist $x(t_k^-)$ and $x(t_k^+)$ $k = 0,1,...,m,$ with $x(t_k^-) = x(t_k)$, $x(0) = \phi\}$, which is a Banach space with the norm

$$\|x\|_\Omega = \max\{\|x_k\|_{J_k},\ k = 0,1,...,m\},$$

Where $x(t_k)$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0,1,...m$, and $\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|_\alpha$.

In this section, we state and prove the existence theorem for the problem $(1.1)-(1.3)$. Let us list the following hypothesis: for some $\alpha \in (0,1)$.

(H_1) There exists a constant $\beta \in (0,1)$ $\alpha \le \beta \le 1$ such that $g : [0,a] \times X_\alpha \to X_\beta$ is a continuous function, and $A^\beta g : [0,a] \times X_\alpha \to X_\alpha$ satisfies the Lipschitz condition, that is, there exists a constants $L > 0$, $L_1 > 0$ such that

$$\|A^\beta g(t,\psi_1) - A^\beta g(t,\psi_2)\|_\alpha \le L(\|\psi_1 - \psi_2\|_B \text{ for } \psi_1, \psi_2 \in B$$

and

$$\left\| A^{\beta} g(t,\psi) \right\| \le L_1(\|\psi\|_{\beta} + 1) \text{ for } \psi \in B,\ t \in J.$$

(H_2) $F : J \times B \to P_{cv,cP}(X)$; $(t,\psi) \to F(t,\psi)$ is measurable with respect to t for each $\psi \in B$, u.s.c with respect to ψ for each $\psi \in J$, and for each fixed $\psi \in B$, the set $S_{F,\psi} = \{ f \in L^1(J,X) : f(t) \in F(t,\psi) \text{ for a.e } t \in J\}$ is non empty.

(H_3) There exists an integrable function $m : J \to [0,+\infty)$ and a continuous nondecreasing function $W : [0,+\infty) \to [0,+\infty)$ such that

$$\|F(t,\psi)\| = \sup\{\|f\| : f(t) \in F(t,\psi)\} \le m(t) W(\|\psi\|_B),\ (t,\psi) \in J \times B.$$

(H_4) The function $I_i : B \to X$ is continuous and there are positive constants $L_i,\ i = 1,2,\dots n$, such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \le L_i \|\psi_1 - \psi_2\|_B,$$

for every $\psi_j \in B,\ j = 1,2.,\ i = 1,2,\dots n.$

Lemma 3.1. [11] If $y : (-\infty, a) \to X$ is a function such that $y_0 = \varphi$ and $y|_J \in \wp C(J,X)$, then

$$\|y_s\|_B \le (M_a + \overline{J^{\varphi}}) \|\varphi\|_B + K_a \sup\{\|y(\theta)\|;\ \theta \in [0, \max\{0,s\}]\},\ s \in \Re(\rho^-) \cup J,$$

where $\overline{J^{\varphi}} = \sup_{t \in \Re(\rho^-)} J^{\varphi}(t),\ M_a = \sup_{t \in J} M(t)$ *and* $K_a = \max_{t \in J} K(t).$

Theorem 3.2. Assume that (H_1)−(H_4) are satisfied, then the problem (1.1)−(1.3) admits at least one mild solution provided that,

$$K_{\alpha}(L\|A^{-\beta}\| + L\frac{C_{1-\beta}}{\beta} a^{\beta} + M\sum_{i=1}^{n} L_i) < 1 \qquad (3.1)$$

$$\|A^{-\beta}\| L_1 K_{\alpha} + \frac{C_{1-\beta}}{\beta} a^{\beta} L_1 K_a + K_a M (Lim_{\xi \to \infty^+} \frac{W(\xi)}{\xi} \int_0^a m(s) ds + \sum_{i=1}^{n} L_i) < 1 \qquad (3.2)$$

Proof. On the space $\Omega = \{u \in PC : u(0) = \phi(0)\}$ endowed with the uniform convergence norm $(\|.\|_{\infty})$, we define the operator $N : \Omega \to P(\Omega)$ by

$$N(x) = \{u \in \Omega : u(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,\overline{x_t}) + \int_0^t AT(t-s) g(s,\overline{x_s}) ds$$

$$+ \int_0^t T(t-s) f(s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(\overline{y}_{t_i}),\ f \in S_{F,\overline{x_{\rho}}},\ t \in J\},$$

where $S_{F,\overline{x_{\rho}}} = \{ f \in L^1(J,X) : f(t) \in F(t, \overline{x}_{(t,x_t)}), t \in J\}$, $\overline{x} : (-\infty, a] \to X$ is such that $\overline{x_0} = \phi$ and $\overline{x} = y$ on J. Let $\overline{\phi} : (-\infty, a) \to X$ be the extension of ϕ to $(-\infty, a]$ such that $\overline{\phi}(\theta) = \phi(0)$

on J and $\tilde{J}^{\phi} = \sup\{J^{\phi}(s): s \in \Re(\rho^{-})\}$. In order to apply [Lemma 2.2] we give the proof in several steps.

Step:1 There exists $r > 0$ such that $N(B_r) \subset B_r$, where $B_r = \{x \in \Omega : \|x\|_{\infty} \le r\}$. For each $r > 0$, B_r is clearly a bounded closed convex subset in Ω. We claim that there exists $r > 0$, such that $N(B_r) \subset B_r$, where $N(B_r) = \bigcup_{x \in B_r} N(x)$. In fact, if it is not true, then for each $r > 0$, there exists $x^r \in B_r$ such that $u^r \in N(x^r)$ but $\|u^r\|_{\infty} > r$ and

$$u^r(t) = T(t)[\phi(0) - g(0,\phi)] + g(t, \overline{x_t^r}) + \int_0^t AT(t-s) g(s, \overline{x_s^r}) ds$$

$$+ \int_0^t T(t-s) f^r(s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(\overline{x_{t_i}^r}),$$

For some $f^r \in S_{F, \bar{x}_P^r}$. Consequently, we have

$$r < \|u^r\|_{\infty} = \max_{t \in J} |u^r(t)|$$

$$= \left\| T(t)[\phi(0) - g(0,\phi)] + g(t, \overline{x_t^r}) + \int_0^t AR(t-s) g(s, \overline{y_s^r}) ds + \int_0^t T(t-s) f^r(s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(\overline{x_{t_i}^r}) \right\|$$

Hence , Lemma 3.1 for some $t \in [0,a]$

$$r \le \left\| T(t)[\phi(0) - A^{-\beta} A^{\beta} g(0,\phi)] \right\| + \left\| A^{-\beta} A^{\beta} g(t, \overline{x_t^r}) \right\| + \left\| \int_0^t A^{1-\beta} T(t-s) g(s, \overline{y_s^r}) ds \right\|$$

$$+ \left\| \int_0^t T(t-s) f^r(s) ds \right\| + \left\| \sum_{0<t_i<t} T(t-t_i) I_i(\overline{x_{t_i}^r}), \right\|$$

$$\le M[\|\phi(0)\| + \|A^{-\beta}\| L_1(\|\phi\|_B + 1) + \|A^{-\beta}\| L_1(\|\overline{x_t^r}\|_B + 1)$$

$$+ \int_0^t \left\| \frac{C_{1-\beta}}{(t-s)^{1-\beta}} L_1(\|x_s^r\|_B + 1) ds \right\| + M \int_0^t \|f^r(s)\| ds + M \sum_{i=1}^n \left\| I_i(\overline{x_{t_i}^r}) - I_i(0) + I_i(0) \right\|.$$

$$\le M[\|\phi(0)\| + \|A^{-\beta}\| L_1(\|\phi\|_B + 1)] + \|A^{-\beta}\| L_1(K_a r + M_a \|\phi\|_B + 1)$$

$$+ \frac{C_{1-\beta}}{\beta} a^{\beta} L_1(K_a r + M_a \|\phi\|_B + 1)$$

$$+ MW((M_a + \tilde{J}^{\phi})\|\phi\|_B + K_a r) \int_0^a m(s) ds + M \sum_{i=1}^n L_i(K_a r + M_a \|\phi\|_B + \|I_i(0)\|),$$

and thus,

$$1 \le (\|A^{-\beta}\| L_1 K_a + \frac{C_{1-\beta}}{\beta} a^{\beta} L_1 K_a + K_a M \lim_{\xi \to +\infty} \frac{W(\xi)}{\xi} \int_0^a m(s) ds + K_a M \sum_{i=1}^n L_i)$$

which contradicts (3.2). Hence there exists $r > 0$ such that $N(B_r) \subset B_r$.

Let $r > 0$ be such that $N(B_r) \subset B_r$. If $x \in B_r$, from Lemma 3.1, it follows that

$$\|\bar{x}_{\rho(t,\bar{x}_t)}\|_B \le r^* := (M_a + \tilde{J}^{\phi})\|\phi\|_B + K_a r. \qquad (3.3)$$

Step:2 $N(x)$ is convex for each $x \in X$. Indeed, if $u_1, u_2 \in N(x)$,
Then there exist $f_1, f_2 \in S_{F,\bar{x}_\rho}$, such that $t \in J$ we have,

$$u_i(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,\overline{x_t}) + \int_0^t AT(t-s)g(s,\overline{x_s})ds$$

$$+\int_0^t T(t-s)f_i(s)ds + \sum_{0<t_i<t} T(t-t_i)I_i(\overline{x_{t_i}}),\ \ i = 1,2.$$

Let $0 \le \lambda \le 1$. Then for each $t \in J$ we have,

$$(\lambda u_1 + (1-\lambda)u_2)(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,\overline{x_t}) + \int_0^t AT(t-s)g(s,\overline{x_s})ds$$

$$+\int_0^t T(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]ds + \sum_{0<t_i<t} T(t-t_i)I_i(\bar{x}_{t_i}).$$

Since $S_{F,\bar{x}_\rho}$ is convex (because F has convex values), $(\lambda u_1 + (1-\lambda)u_2) \in N(x)$.

Step:3 $N(x)$ is closed for each $x \in X$.
Let $\{x_n\}_{n\ge 0} \in N(x)$ such that $x_n \to x$ in X. Then $x \in X$ and there exists $f_n \in S_{F,\overline{f_\rho}}$
Such that, for each $t \in J$,

$$x_n(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,\overline{x_t}) + \int_0^t AT(t-s)g(s,\overline{x_s})ds$$

$$+\int_0^t T(t-s)f_n(s)ds + \sum_{0<t_i<t} T(t-t_i)I_i(\bar{x}_{t_i}).$$

Using the fact that F has a nonempty compact value, there is a subsequence if necessary to get that v_n converges to v in $L^1(J,X)$ and hence $v \in S_{F,\overline{x_\rho}}$. Then for each $t \in J$,

$$x_n(t) \to x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,\overline{x_t}) + \int_0^t AT(t-s)g(s,\overline{x_s})ds$$

$$+\int_0^t T(t-s)f(s)ds + \sum_{0<t_i<t} T(t-t_i)I_i(\bar{x}_{t_i}).$$

Thus, $x \in N(x)$.

Step: 4 N u.s.c and condensing.

To prove that N is u.s.c and condensing, we introduce the decomposition $N = N_1 + N_2$, where

$$(N_1 y)(t) = g(t,\bar{x}_t) - T(t)g(0,\phi) + \int_0^t AT(t-s)g(s,\bar{x}_s)ds + \sum_{0<t_i<t} T(t-t_i)I_i(\bar{x}_{t_i}).$$

$$(N_2 x)(t) = \{u \in X : u(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s)ds,\ \ f \in S_{F.\bar{x}_\rho}\}$$

we will verify that N_1 is a constraction while N_2 is a completely continuous operator. To prove that N_1 is a contraction, we take $x^*, x^{**} \in B_r$ arbitrarily. Then for each $t \in J$, we have that,

$$\|(N_1x^*)(t)-(N_1x^{**})(t)\| \le \|g(t,\overline{x_t^*})-g(t,\overline{x_t^{**}})\|$$

$$+\left\|\int_0^t AT(t-s)[g(s,\overline{x_s^*})-g(s,\overline{x_s^{**}})]ds\right\|$$

$$+\left\|\sum_{0<t_i<t} T(t-t_i)(I_i(\overline{x_{t_i}^*}-I_i(\overline{x_{t_i}^{**}}))\right\|.$$

$$\le \|A^{-\beta}[A^\beta g(t,\overline{x_t^*})-A^\beta(g(t,\overline{x_t^{**}})]\|$$

$$+\left\|\int_0^t A^{1-\beta}T(t-s)A^\beta[g(s,\overline{x_t^*})-g(s,\overline{x_t^{**}})]ds\right\|$$

$$+M\sum_{i=1}^n L_i\|\overline{x_t^*}-\overline{x_t^{**}}\|_B$$

$$\le L_i\|\overline{x_t^*}-\overline{x_t^{**}}\|_B\|A^{-\beta}\|$$

$$+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L\|\overline{x_s^*}-\overline{x_s^{**}}\|_B\,ds+M\sum_{i=1}^n L_i\|\overline{x_t^*}-\overline{x_t^{**}}\|_B.$$

$$\le L\|A^{-\beta}\|K_a\sup\{\|\overline{x^*}(\theta)-\overline{x^{**}}(\theta)\|,\ 0\le\theta\le s\}$$

$$+L\frac{C_{1-\beta}}{\beta}a^\beta K_a\sup\{\|\overline{x^*}(\theta)-\overline{x^{**}}(\theta)\|,\ 0\le\theta\le s\}$$

$$+M\sum_{i=1}^n L_iK_a\sup\{\|\overline{x^*}(\theta)-\overline{x^{**}}(\theta)\|,\ 0\le\theta\le t\}$$

$$\le L^*\sup_{0\le s\le a}\|\overline{x^*}(s)-\overline{x^{**}}(s)\|$$

$$=L^*\sup_{0\le s\le a}\|x^*(s)-x^{**}(s)\| \quad (Since\ \overline{x}=x\ on\ J),$$

Where

$$L^*=K_a(L\|A^{-\beta}\|+\frac{C_{1-\beta}}{\beta}a^\beta L+M\sum_{i=1}^n L_i),$$

thus,

$$\|(N_1x^*-N_1x^{**}\|_{PC}\le L^*\|x^*-x^{**}\|_{PC}.$$

Therefore, by (3.1) we obtain that N_1 is a contraction,

Next, we show that N_2 is u.s.c

(i) $N_2(B_r)$ is clearly bounded,

(ii) $N_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in J,\ t_1 < t_2$. Let $x \in B_r$ and $u \in N_2(x)$. Than there exists $f \in S_{F,\overline{x_\rho}}$ such that for each $t \in J$, we have

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s)ds.$$

Then,

$$\|u(t_2) - u(t_1)\| \le \|[T(t_2) - T(t_1)]\phi(0)\| + \left\| \int_0^{t_1-\in} [T(t_2 - s) - T(t_1 - s)] f(s)ds \right\|$$

$$+ \left\| \int_{t_1-\in}^{t_1} [T(t_2 - s) - T(t_1 - s)] f(s)ds \right\| + \left\| \int_{t_1}^{t_2} [T(t_2 - s) f(s)ds \right\|$$

$$\le \|T(t_2) - T(t_1)]\phi(0)\| + W(r^*) \int_0^{t_1-\in} \|[T(t_2 - s) - T(t_1 - s)]\| m(s)ds$$

$$+ 2MW(r^*) \int_{t_1-\in}^{t_1} m(s)ds + MW(r^*) \int_{t_1}^{t_2} m(s)ds,$$

Where r^* is defined in (3.3).

As $t_2 \to t_1$ and for $\in$ sufficiently small, the right-hand side of the above ineauality tends to zero independently of $x \in B_r$, since $T(t-s)$ strongly continuous and compactness of $T(t-s),\ t > s$ implies the continuity in the uniform operator topology.

(iii) $(N_2 B_r)(t) = \{u(t) : u \in N_2(B_r),\ t \in J\}$ is precompact for each $t \in J$. Obviously, $N_2(B_r)(t)$ is relatively compact in X for $t = 0$. Let $0 < t \le a$ be fixed and $0 < \in < t$, for $x \in B_r$ and $u \in N_2(x)$, there exists a function $f \in S_{F,\overline{x_\rho}}$ such that

$$u(t) \le T(t)\phi(0) + \int_0^{t-\in} T(t-s)f(s)ds + \int_{t-\in}^{t} T(t-s)f(s)ds.$$

Define,

$$u_\in(t) \le T(t)\phi(0) + \int_0^{t-\in} T(t-s)f(s)ds$$

$$= T(t)\phi(0) + T(t, t-\in) \int_0^{t-\in} T(t-\in, s)f(s)ds$$

Since $T(t-s)(t > s)$ is compact, the set

$$U_\in(t) = \{u_\in(t) : u \in N_2(B_r)\}$$

Is relatively compact in X for every $\in$, $0 < \in < t$. Moreover, for every $u \in N_2(B_r)$

$$\|u(t) - u_\in(t)\| = \left\| \int_{t-\in}^{t} T(t-s)f(s)ds \right\|$$

$$\leq MW(r^*)\int_{t-\epsilon}^{t} m(s)ds \to 0,$$

as $\epsilon \to 0$, where r^* is defined in (3.3). We note that there are relatively compact sets arbitrarily close to the set $\{u(t): u \in N_2(B_r)\}$..

So the set $\{u(t): u \in N_2(B_r)\}$ is relatively compact in X.

From the Arzela-Ascoli theorem we can conclude that N_2 is a completely continuous multi-valued map.

(iv) N_2 has a closed graph.

Let $x^n \to x^*, x^n \in B_r, u_n \in N_2(x^n)$, and $u_n \to u^*$, we prove that $u^* \in N_2(x^*)$. The relation $u_n \in N_2(x^n)$ means that there exists $f_n \in S_{F,\overline{x_\rho^n}}$ such that for each $t \in J$.

$$u_n(t) = T(t)\phi(0) + \int_0^t T(t-s) f_n(s)ds.$$

We must prove that there exists $f^* \in S_{F,\overline{x_\rho^n}}$ such that for each $t \in J$,

$$u^*(t) = T(t)\phi(0) + \int_0^t T(t-s) f^*(s)ds.$$

we have,

$$\left\|[u_n - T(t)\phi(0)] - [u^* - T(t)\phi(0)]\right\|_{\wp C} \to 0$$

Consider the linear continuous operator

$$\Gamma^* : L^1(J,X) \to C(J,X), \quad f \to \Gamma^*(f)(t) = \int_0^t T(t-s) f(s)ds.$$

From Lemma 2.1, it follows that $N^* o S_F$ is a closed graph operator. Moreover, we have

$$u_n(t) - T(t)\phi(0) \in \Gamma^*(S_{F,\overline{x_\rho^n}})$$

In view of $x^n \to x^*$, it follows from Lemma 2.1 again that

$$u^*(t) - T(t)\phi(0) \in \Gamma^*(S_{F,\overline{x_\rho^n}})$$

that is, there must exist a $x^*(t) \in S_{F,\overline{x_\rho^n}}$ such that

$$u^*(t) - T(t)\phi(0) = \Gamma^*(f^*(t)) = \int_0^t T(t-s) f^*(s)ds.$$

Therefore, N_2 is u.s.c. Hence $N = N_1 + N_2$ is u.s.c. and condensing. By the fixed point Lemma 2.2, there exists a fixed point x for N on B_r, which implies that the problem $(1.1)-(1.3)$ has a mild solution.

Example

In this section, we consider an application of our abstract results. At first we introduce the required technical framework. In the rest of this section, $X = L^2([0,\pi])$ and A be the operator $Au = u^{''}$ with domain $D(A) = \{u \in X : u^{''} \in X, u(0) = u(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup on X. Furthermore, A has a discrete spectrum with eigen values of the form $-n^2$, $n \in N$, whose corresponding (normalized) eigen function are given by $z_n(\zeta) = \sqrt{\frac{2}{\pi}}\sin(n\zeta)$. In addition, the following properties hold.

(a) $\{z_n : n \in N\}$ is an orthonormal basis of X :

(b) For $u \in X, T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} < u, z_n > z_n$ and $Au = -\sum_{n=1}^{\infty} n^2 < u, z_n > z_n$, for $u \in D(A)$:

(c) It is possible to define the fractional power $(-A)^{\alpha}, \alpha \in (0,1)$, as a closed linear operator over its domain $D((-A)^{\alpha})$. More precisely, the operator $(-A)^{\alpha} : D((-A)^{\alpha}) \subseteq X \to X$

Is given by $(-A)^{\alpha} u = \sum_{n=1}^{\infty} n^{2\alpha} < u, z_n > z_n$,

For all $u \in D(-A)^{\alpha}$,

where $D(-A)^{\alpha} = \{u \in X : \sum_{n=1}^{\infty} n^{2\alpha} < u, z_n > z_n \in X\}$;

(d) If X_{α} is the space $D(-A)^{\alpha}$ endowed with the graph norm $\|.\|_{\alpha}$ then X_{α} is a Banach space. Moreover, for $0 \le \beta \le \alpha \le 1$, $X_{\alpha} \subset X_{\beta}$; the inclusion $X_{\alpha} \to X_{\beta}$ is completely continuous and there constants $C_{\alpha} > 0$ such that $\|T(t)\|_{L(X_{\alpha};X)} \le \frac{C_{\alpha}}{t^{\alpha}}$ for $t \ge 0$.

As an application of the theorem (3.1), we study the following impulsive partial neutral functional differential system.

$$\frac{\partial}{\partial t}[u(t,\zeta) - \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,s)u(s,\eta)d\eta ds] \in \int_{-\infty}^{t} a(s-t)u(s-\rho_1(t)\rho_2(\|u(t)\|),\eta)ds,$$

$$t \in I,\ \eta \in [0,\pi] \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0,\ t \in I \tag{4.2}$$

$$u(\tau,\zeta) = \phi(\tau,\zeta),\ \tau \le 0,\ 0 \le \zeta \le \pi, \tag{4.3}$$

$$u(t_i^+) - u(t_i^-) = I_i(u(t_i)),\ i = 1,2,....,m, \tag{4.4}$$

where $\phi \in B = \wp C \times L^2(g,x)$ and $0 < t_1 < \cdot \cdot \cdot < t_n < b$ are prefixed. Under these conditions, we can define the operators, ρ, g, F : I × B → X and I_i : B → X by,

$$\rho(t,\psi) = \rho_1(t)\rho_2(\|\psi(0)\|),$$

$$g(\psi)(\zeta)) = \int_{-\infty}^{0}\int_{0}^{\pi} b(s,v,\zeta)\psi(s,v)dvds,$$

$$F(\psi)(\zeta) = \int_{-\infty}^{0} a(s)\psi(s,\zeta)ds,$$

$$u(t_i^+) - u(t_i^-) = I_i(u(t_i)),\ i = 1,2,...,m,$$

which permit to transform system (4.1) − (4.4) into the system (1.1) − (1.3). Moreover, the maps, $g, F, I_i, i = 1,2,..., m$, are bounded linear operators. Thus, the assumptions (H1) − (H4) are hold. All conditions of theorem 3.1 are now fulfilled so we deduce that (4.1) − (4.4) has an integral solution.

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