PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

[1]Dr. Balvir Singh, [2] Arun Saini

[1] Assoc. Professor, [2] Assistant Professor

[1] Department of Mathematics

R P Degree College, Kamalganj, Farrukhabad-209601

(U.P.) India

Abstract : Let $f_m(z)=z+\sum_{k=2}^{m}a_kz^k$ be the sequence of partial sums of a function $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ that is analytic in $|z|<1$ and belong to the class $S_n(\alpha)$, where $(0\le\alpha<1)$. When the coefficients of $\{a_k\}$ are " small " we determine sharp lower bounds for $\text{Re}\left\{\frac{D^p f(z)}{D^p f_m(z)}\right\}$ and $\text{Re}\left\{\frac{D^p f_m(z)}{D^p f(z)}\right\}$, where D^p stands for the Salagean derivative introduced in [4].

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I. INTRODUCTION

Let A denote the class of functions f of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \qquad (1.1)$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. Further T denotes subclass of A consisting of functions $f(z)$ of the form

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k, \qquad a_k\ge 0. \qquad (1.2)$$

We denote by $S^*(\alpha), K(\alpha), (0\le\alpha<1)$, the class of starlike functions of order α and class of convex functions of order α, respectively, where

$$S^*(\alpha)=\left\{f\in S:\text{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha, z\in U\right\}$$

and

$$K(\alpha)=\left\{f\in S:\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, z\in U\right\}.$$

We also denote by $T^*(\alpha)$ and $C(\alpha)$ the subclasses of T that are, respectively, starlike of order α and convex of order α.

For $f(z)$ belonging to A, Salagean [4] has introduced the following operator called the Salagean operator

$$D^0 f(z)=f(z)$$

$$D^1 f(z) = zf'(z)$$

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \qquad \left(n \in N = \{1,2,3.....\}\right).$$

Note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \qquad \left(n \in N_0 = N \cup \{0\}\right).$$

A function $f(z) \in A$ is said to belong to the class $S_n(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \alpha, \qquad (z \in U) \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$ and $n \in N_0$.

The class $S_n(\alpha)$ have been studied by various authors (for example [1], [2], [3]).

Note that

$S_0(\alpha) = S^*(\alpha)$ and $S_1(\alpha) = K(\alpha)$.

A sufficient condition for a function f of the form (1.1) to be in $S_n(\alpha)$ is that

$$\sum_{k=2}^{\infty} \frac{k^n (k-\alpha)}{1-\alpha} |a_k| \le 1. \tag{1.4}$$

For the functions of the form (1.2) the sufficient condition (1.4) is also necessary. For detailed study see [1].

In the present paper, we determine sharp lower bounds for $\operatorname{Re}\left\{\frac{D^p f(z)}{D^p f_m(z)}\right\}$ and $\operatorname{Re}\left\{\frac{D^p f_m(z)}{D^p f(z)}\right\}$ (where $f_m(z) = z + \sum_{k=2}^{m} a_k z^k$ is the sequence of partial sums of $f(z)$ given by (1.1) and coefficients of f are sufficiently small to satisfy the condition (1.4)) which are motivated from the investigation of Silverman [5].

2. Main Results

Theorem 2.1: If f of the form (1.1) satisfies the condition (1.4), then

$$\operatorname{Re}\left\{\frac{D^p f(z)}{D^p f_m(z)}\right\} \ge \frac{(m+1)^{n-p}(m+1-\alpha) - (1-\alpha)}{(m+1)^{n-p}(m+1-\alpha)}, \qquad (z \in U), \tag{2.1}$$

and

$$\operatorname{Re}\left\{\frac{D^p f_m(z)}{D^p f(z)}\right\} \ge \frac{(m+1)^{n-p}(m+1-\alpha)}{(m+1)^{n-p}(m+1-\alpha) + (1-\alpha)}, \qquad (z \in U). \tag{2.2}$$

The results (2.1) and (2.2) are sharp for every m with the function given by

$$f(z) = z + \frac{1-\alpha}{(m+1)^n (m+1-\alpha)} z^{m+1}. \tag{2.3}$$

where $0 \le \alpha < 1, n \in N_0$ and $p \le n+1$.

Proof: Define the function $\omega(z)$ by

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\left[\frac{D^p f(z)}{D^p f_m(z)}-\frac{(m+1)^{n-p}(m+1-\alpha)-(1-\alpha)}{(m+1)^{n-p}(m+1-\alpha)}\right]$$

$$=\frac{1+\sum_{k=2}^{m}k^p a_k z^{k-1}+\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p a_k z^{k-1}}{1+\sum_{k=2}^{m}k^p a_k z^{k-1}}. \qquad (2.4)$$

It suffices to show that $|\omega(z)|\le 1$. Now, from (2.4) we can write

$$\omega(z)=\frac{\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p a_k z^{k-1}}{2+2\sum_{k=2}^{m}k^p a_k z^{k-1}+\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p a_k z^{k-1}}.$$

Hence we obtain

$$|\omega(z)|\le\frac{\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p |a_k|}{2-2\sum_{k=2}^{m}k^p |a_k|-\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p |a_k|}.$$

Now $|\omega(z)|\le 1$ if

$$2\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p |a_k|\le 2-2\sum_{k=2}^{m}k^p |a_k|$$

or, equivalently,

$$\sum_{k=2}^{m}k^p |a_k|+\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p |a_k|\le 1. \qquad (2.5)$$

From the condition (1.4), it is sufficient to show that

$$\sum_{k=2}^{m}k^p |a_k|+\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p |a_k|\le\sum_{k=2}^{\infty}\frac{k^n(k-\alpha)}{1-\alpha}|a_k|$$

which is equivalent to

$$\sum_{k=2}^{m}\frac{k^{n+1}-\alpha k^n-k^p(1-\alpha)}{1-\alpha}|a_k|+\sum_{k=m+1}^{\infty}\frac{k^{n+1}-\alpha k^n-(m+1)^{n-p}(m+1-\alpha)k^p}{1-\alpha}|a_k|\ge 0.$$

To see that the function given by (2.3) gives the sharp result, we observe that for $z=re^{i\pi/m}$ that

$$\frac{D^p f(z)}{D^p f_m(z)}=1+\frac{1-\alpha}{(m+1)^n(m+1-\alpha)}(m+1)^p z^m\to 1-\frac{1-\alpha}{(m+1)^{n-p}(m+1-\alpha)}$$

$$=\frac{(m+1)^{n-p}(m+1-\alpha)-(1-\alpha)}{(m+1)^{n-p}(m+1-\alpha)},\quad \text{when } r\to 1^-.$$

To prove the second part of this theorem, we may write

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{(m+1)^{n-p}(m+1-\alpha)+(1-\alpha)}{1-\alpha}\left[\frac{D^p f_m(z)}{D^p f(z)}-\frac{(m+1)^{n-p}(m+1-\alpha)}{(m+1)^{n-p}(m+1-\alpha)+(1-\alpha)}\right]$$

$$=\frac{1+\sum_{k=2}^{m}k^p a_k z^{k-1}-\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p a_k z^{k-1}}{1+\sum_{k=2}^{\infty}k^p a_k z^{k-1}}$$

where

$$|\omega(z)|\le\frac{\frac{(m+1)^{n-p}(m+1-\alpha)+(1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p|a_k|}{2-2\sum_{k=2}^{m}k^p|a_k|-\left\{\frac{(m+1)^{n-p}(m+1-\alpha)-(1-\alpha)}{1-\alpha}\right\}\sum_{k=m+1}^{\infty}k^p|a_k|}\le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m}k^p|a_k|+\frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha}\sum_{k=m+1}^{\infty}k^p|a_k|\le 1. \tag{2.6}$$

Since the L.H.S. of (2.6) is bounded above by $\sum_{k=2}^{\infty}\frac{k^n(k-\alpha)}{1-\alpha}|a_k|$, and the proof is complete. Finally, equality holds in (2.2) for the function given in (2.3).

Taking $n=0, p=0$ in Theorem 2.1, we obtain the following result given by Silverman in [5].

Corollary 2.2 ([5]). If f of the form (1.1) and satisfies the condition $\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k|\le 1$, then

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\}\ge\frac{m}{(m+1-\alpha)},\qquad (z\in U), \tag{2.7}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\}\ge\frac{m+1-\alpha}{m+2-2\alpha},\qquad (z\in U). \tag{2.8}$$

The results are sharp with the function given by

$$f(z)=z+\frac{1-\alpha}{(m+1-\alpha)}z^{m+1}. \tag{2.9}$$

Taking $n=0, p=1$ in Theorem 2.1, we obtain the following result given by Silverman in [5].

Corollary 2.3 ([5]) If f of the form (1.1) and satisfies the condition $\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k|\le 1$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_m(z)}\right\}\ge\frac{m\alpha}{m+1-\alpha},\qquad (z\in U), \tag{2.10}$$

and

$$\operatorname{Re}\left\{\frac{f_m'(z)}{f'(z)}\right\} \geq \frac{m+1-\alpha}{(m+1)(2-\alpha)-\alpha}, \qquad (z \in U). \qquad (2.11)$$

The results are sharp with the function given by (2.9).

Taking $n=1, p=0$ in Theorem 2.1, we obtain the following result given by Silverman in [5].

Corollary 2.4 ([5]) If f of the form (1.1) and satisfies the condition $\sum_{k=2}^{\infty} \frac{k(k-\alpha)}{1-\alpha}|a_k| \leq 1$, then

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}, \qquad (z \in U), \qquad (2.12)$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{(m+1)(m+1-\alpha)}{(m+1)(m+1-\alpha)+(1-\alpha)}, \qquad (z \in U). \qquad (2.13)$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(m+1)(m+1-\alpha)} z^{m+1}. \qquad (2.14)$$

Taking $n=1, p=1$ in Theorem 2.1, we obtain the following result given by Silverman in [5].

Corollary 2.5 ([5]) If f of the form (1.1) and satisfies the condition $\sum_{k=2}^{\infty} \frac{k(k-\alpha)}{1-\alpha}|a_k| \leq 1$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_m'(z)}\right\} \geq \frac{m}{m+1-\alpha}, \qquad (z \in U), \qquad (2.15)$$

and

$$\operatorname{Re}\left\{\frac{f_m'(z)}{f'(z)}\right\} \geq \frac{m+1-\alpha}{m+2-2\alpha}, \qquad (z \in U). \qquad (2.16)$$

The results are sharp with the function given by (2.14).

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