FIXED POINT THEOREMS IN A COMPLETE B-METRIC SPACE FOR TWO SELF MAPS

[1]Dr. Giniswamy, [2]Venkatesh Bhatt, [3]Jeyanthi C.

[1] Associative Professor and Head, [2]Research Scholar, [3]Assistant Professor
[1]Department of Mathematics,
[1]PES College of Science, Arts and Commerce, Mandya, Karnataka, India-571401

Abstract: In this paper we obtain a common fixed point result for two self maps in a complete b-metric space which improve and generalize the result established by Swati et. al.[8] (2016).

Index Terms **- fixed point, b-metric space.**

I. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle [2] is a fundamental result in the fixed point theory, which has been extended in many different ways. Also, there are several generalizations of usual metric spaces. One of the generalizations of metric spaces is b-metric spaces introduced in 1989 by Bakhtin [1] and proved some fixed point theorems for some contractive mappings in b-metric spaces which are also generalizations of Banach's contraction principle in metric spaces. After this, many authors presented papers on fixed point theory on a single valued and multi valued operators in b-metric spaces [4, 5, 6, 7]. In the present paper we prove a fixed point theorem in complete b-metric space for two self maps.

Definition 1.1 [1]. Let **X** be a non empty set, $s \geq 1$ be a given real number and $d: X \times X \rightarrow R_+$ be a function. We say d is a b-metric on X if and only if for all x, y and z in X the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$

A pair (X, d) is called a b-metric space. If s =1, b-metric reduces to usual metric.

Definition 1.2 [3] Let **(X, d)** be a b-metric space. Then a sequence $\{x\}^{\infty}_{n=1}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 1.3 [3] Let **(X, d)** be a b-metric space. Then a sequence $\{x\}^{\infty}_{n=1}$ in X is called a convergent sequence if and only if there exists $x \in X$ such that for all $n \in N$ and $n > n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$, then we write $\lim_{n\to\infty} x_n = x$.

Definition 1.4 [3] The b-metric is complete if every Cauchy sequence is convergent.

II MAIN RESULT

Theorem 2.1. Let (X, d) be a complete b-metric space and let S,T: X→X be two self mappings such that

$$d(Sx, Ty) \leq h \max\{d(x, y), d(x, Sx), d(y, Ty), d(x,Ty), d(y, Sx)\} \quad (1)$$

with $0 < h < 1$ then S and T have unique common fixed point.

Proof :- Let $x_0 \in X$ and $\{x\}^{\infty}_{n=1}$ be a sequence in X such that

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1} \quad (2)$$

for n = 0, 1, 2,...

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq h \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\} \\ &\leq h \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} \\ &\leq h \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\} \end{aligned}$$

Case 1: If $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\} = d(x_{2n}, x_{2n+1})$ then

$$d(x_{2n+1}, x_{2n+2}) \leq hs\, d(x_{2n}, x_{2n+1}).$$

Similarly, $d(x_{2n}, x_{2n+1}) \leq hs\, d(x_{2n-1}, x_{2n})$.

Continuing the process, we get $d(x_{2n+1}, x_{2n+2}) \leq (hs)^{2n+1} d(x_0, x_1)$.

Case 2: If $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\} = d(x_{2n}, x_{2n+2})$ then

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq hs\, d(x_{2n}, x_{2n+2}) \\ &\leq hs[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &\leq \frac{hs}{1-hs} d(x_{2n}, x_{2n+1}) \end{aligned}$$

$$d(x_{2n+1}, x_{2n+2}) \leq k\, d(x_{2n}, x_{2n+1}), \quad \text{where } k = \frac{hs}{1-hs} < 1.$$

Continuing this process, we get $d(x_{2n+1}, x_{2n+2}) \leq k^{2n+1} d(x_0, x_1)$.

Case 3: If $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$ then

$$d(x_{2n+1}, x_{2n+2}) \leq hs[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

Thus S and T are contractive mappings.

Now we show that $\{x_n\}$ is Cauchy sequence in X. Let $m, n \in N$, $m > n$

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq s[d(x_n, x_{n+1})] + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\leq s(hs)^n [d(x_0, x_1)] + s^2(hs)^{n+1}[d(x_0, x_1)] + \ldots \\ &\leq s(hs)^n [d(x_0, x_1)][1 + s^2h + (s^2h)^2 + \ldots + (s^2h)^{(m-n-1)}] \end{aligned}$$

$$d(x_n, x_m) \leq s(hs)^n \left[\frac{1-(s^2h)^{m-n}}{1-s^2h}\right] d(x_0, x_1)$$

Then, as $m, n \to \infty$, $\lim_{n\to\infty} s(hs)^n \left[\frac{1-(s^2h)^{m-n}}{1-s^2h}\right] d(x_0, x_1) = 0$

Hence $\{x_n\}$ is Cauchy sequence in X. So $\{x_n\}$ converges to $x^* \in X$.

Now we show that x^* is a fixed point of T.

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_{2n+1}) + d(x_{2n+1}, Tx^*)] \\ &\leq s[d(x^*, x_{2n+1}) + d(Sx_{2n}, Tx^*)] \\ &\leq s[d(x^*, x_{2n+1}) + s[h \max\{d(x_{2n}, x^*), d(x_{2n}, Sx_{2n}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, Sx_{2n})\}] \\ &\leq s[d(x^*, x_{2n+1}) + sh[\max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, x_{2n+1})\}]. \end{aligned}$$

Now different cases arises

Case 1: If $\max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, x_{2n+1})\} = d(x_{2n}, x^*)$ then

$$d(x^*, Tx^*) \leq s d(x^*, x_{2n+1}) + sh\, d(x_{2n}, x^*),$$

as $n\to\infty$, $d(x^*, Tx^*) = 0$. Hence x^* is the fixed point of T.

Case 2: If $\max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$ then

$$d(x^*, Tx^*) \leq s\, d(x^*, x_{2n+1}) + sh\, d(x_{2n}, x_{2n+1}),$$

as $n\to\infty$, $d(x^*, Tx^*) = 0$. Hence x^* is the fixed point of T.

Case 3: If $\max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, x_{2n+1})\} = d(x_{2n}, Tx^*)$ then

$$\begin{aligned} d(x^*, Tx^*) &\leq s\, d(x^*, x_{2n+1}) + sh\, d(x_{2n}, Tx^*) \\ &\leq s\, d(x^*, x_{2n+1}) + s^2h[d(x_{2n}, x^*) + d(x^*, Tx^*)] \end{aligned}$$

$$(1-hs^2)\, d(x^*, Tx^*) \leq s\, d(x^*, x_{2n+1}) + s^2h\, d(x_{2n}, x^*)$$

$$d(x^*, Tx^*) \leq \frac{s}{(1-hs^2)} d(x^*, x_{2n+1}) + \frac{hs^2}{(1-hs^2)} d(x_{2n}, x^*)$$

as $n\to\infty$, $d(x^*, Tx^*) = 0$. Hence x^* is the fixed point of T.

Case 4: If $\max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, x_{2n+1})\} = d(x^*, x_{2n+1})$ then

$$\begin{aligned} d(x^*, Tx^*) &\leq s\, d(x^*, x_{2n+1}) + sh\, d(x^*, x_{2n+1}) \\ &\leq (s+sh) d(x^*, x_{2n+1}). \end{aligned}$$

as $n\to\infty$, $d(x^*, Tx^*) = 0$. Hence x^* is the fixed point of T.

Case 5: If $\max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Tx^*), d(x_{2n}, Tx^*), d(x^*, x_{2n+1})\} = d(x^*, Tx^*)$ then

$$d(x^*, Tx^*) \leq s\, d(x^*, x_{2n+1}) + sh\, d(x^*, Tx^*)$$

$$(1-sh)\, d(x^*, Tx^*) \leq s\, d(x^*, x_{2n+1}),$$

as $n\to\infty$, $d(x^*, Tx^*) = 0$. Hence x^* is the fixed point of T.

Now we show x* is fixed point of S.

$$d(x^*, Sx^*) \leq s[d(x^*, x_{2n+2}) + d(x_{2n+2}, Sx^*)]$$
$$\leq s[d(x^*, x_{2n+2}) + d(Sx^*, Tx_{2n+1})]$$
$$\leq s[d(x^*, x_{2n+2})] + s[h \max\{d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, T x_{2n+1}), d(x^*, T x_{2n+1}), d(x_{2n+1}, Sx^*)\}]$$
$$\leq s[d(x^*, x_{2n+1})] + sh [\max\{ d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, x_{2n+2}), d(x^*, x_{2n+2}), d(x_{2n+1}, Sx^*)\}].$$

Now different cases arise,

Case 1: If $\max \{d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, x_{2n+2}), d(x^*, x_{2n+2}), d(x_{2n+1}, Sx^*)\} = d(x^*, x_{2n+1})$ then

$$d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2}) + sh\, d(x^*, x_{2n+1})$$

as $n\to\infty$, $d(x^*, Sx^*) = 0$. Hence x* is the fixed point of S.

Case 2: If $\max\{d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, x_{2n+2}), d(x^*, x_{2n+2}), d(x_{2n+1}, Sx^*)\} = d(x_{2n+1}, x_{2n+2})$ then

$$d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2}) + sh\, d(x_{2n+1}, x_{2n+2})]$$

as $n\to\infty$, $d(x^*, Sx^*) = 0$. Hence x* is the fixed point of S.

Case 3: If $\max\{d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, x_{2n+2}), d(x^*, x_{2n+2}), d(x_{2n+1}, Sx^*)\} = d(x_{2n+1}, Sx^*)$ then

$$d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2}) + sh\, d(x_{2n+1}, Sx^*)$$
$$(1- sh)\, d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2})$$
$$d(x^*, Sx^*) \leq \frac{s}{(1-sh)}\, d(x^*, x_{2n+2}),$$ as $n\to\infty$, $d(x^*, Sx^*) = 0$. Hence x* is the fixed point of S.

Case 4: If $\max\{d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, x_{2n+2}), d(x^*, x_{2n+2}), d(x_{2n+1}, Sx^*)\} = d(x^*, x_{2n+2})$ then

$$d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2}) + sh\, d(x^*, x_{2n+2})$$
$$\leq (s + sh)\, d(x^*, x_{2n+2})$$

as $n\to\infty$, $d(x^*, Sx^*) = 0$. Hence x* is the fixed point of S.

Case 5: If $\max\{d(x^*, x_{2n+1}), d(x^*, Sx^*), d(x_{2n+1}, x_{2n+2}), d(x^*, x_{2n+2}), d(x_{2n+1}, Sx^*)\} = d(x^*, Sx^*)$ then

$$d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2}) + sh\, d(x^*, Sx^*)$$
$$(1- sh)\, d(x^*, Sx^*) \leq s\, d(x^*, x_{2n+2})$$

as $n\to\infty$, $d(x^*, Sx^*) = 0$. Hence x* is the fixed point of S.

Therefore x* is a common fixed point of S and T.

Uniqueness :

Now we show that x* is unique fixed point of S and T.

Assume x** is another fixed point of S and T then we have, $Sx^{**} = x^{**}$ and $Tx^{**} = x^{**}$

$$d(x^{**}, x^*) = d(Sx^*, Tx^{**})$$
$$\leq h \max\{d(x^*, x^{**}), d(x^*, Sx^*), d(x^{**}, Tx^{**}), d(x^*, T x^{**}), d(x^{**}, Sx^*)\}$$
$$\leq h \max\{d(x^*, x^{**}), d(x^*, x^{**}), d(x^*, x^{**})\}$$
$$\leq h\, d(x^*, x^{**})$$
$$(1- h)\, d(x^{**}, x^*) \leq 0$$

Therefore $d(x^*, x^{**}) = 0$. Implies $x^* = x^{**}$. Hence x* is the unique fixed point of S and T.

Example :- Let X = [0, 1] and $d: X \times X \to X$ defined by $d(x, y) = |x-y|$ for all x, y in X then (X, d) is b-metric space with s=2.Now we define two self maps $S, T: X \to X$ such that $Sx = \frac{x}{4}$ and $Ty = \frac{y}{2}$. S and T satisfies the contractive conditions of theorem 2.1 and x = 0 is unique common fixed point of S and T.

Corollary:- Let (X ,d) be a complete b-metric space and let $T: X\to X$ be a self mapping such that $d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, with $0 < h < 1$ then T has unique common fixed point.

Proof :By putting S = T in theorem 2.1 we get the corollary.

Example:- Let X = [0, 1] and $d: X \times X \to X$ defined by $d(x, y) = |x-y|$ for all x, y in X then (X, d) is b-metric space with s=2. Now we define a self map $T: X \to X$ such that $Tx = \frac{x}{4}$ which satisfies the contractive conditions of theorem 2.1 and x = 0 is unique common fixed point of T.

References

[1] Bakhtin, I. A .1989. The contraction mapping principle in almost metric spaces, Funct. Anal.30, Unianowsk, Gos. Ped. Inst. : 26-37.

[2] Banach, S. 1922. Surles operations dans les ensembles abstract et leur application aux equation integrals. Fund. Math., 3:133-181.

[3] Boriceanu, M. 2009. Fixed point theory for multivalued generalized contraction on a set with two b-metric, studia. Univ. Babes, Bolya: Math., Liv(3):1-14.

[4] Boriceanu, M. Bota, M. and Petrusel, A. 2010. Mutivalued fractals in b-metric spaces, Cent. Eur. J. Math.8(2), 367-377 .

[5] Bota, M. Molnar, A. and Varga, C. 2011. On ekeland's variational principle in b-metric spaces. Fixed Point Theory, 12, (2): 21-28.

[6] Czerwik, S. 1993. Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis, 1: 5–11.

[7] Czerwik.S,1998. Nonlinear set-valued contraction mappings in b-metric spaces," Atti sem math fiq Univ.Modena, 46(2): 263- 276

[8] Swati Agrawal, K. Qureshi and Jyoti Nema. 2016. A Fixed Point Theorem for b-Metric Space. International Journal of Pure and Applied Mathematical Sciences. ISSN 0972-9828, 9(1): 45-50.