ON A NEW SUBCLASS OF A HARMONIC UNIVALENT FUNCTIONS

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Abstract : In the present paper, we introduce and study a new subclass of Salagean-type harmonic univalent functions and obtain results regarding coefficient inequalities, growth and distortion theorems for this class.

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I. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that

$$|h'(z)| > |g'(z)|,\ z \in D.$$

Let S_H denotes the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \ ,\ g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \qquad (1.1)$$

Clunie and Sheil-Small [2] investigated the class S_H as well as its geometric subclasses and established some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. In fact, by introducing new subclasses Avci and Zlotkiewicz [1], Silverman [15], Silverman and Silvia [16], Jahangiri [7], Jahangiri et al. [8], Dixit and Porwal [3]-[5], Porwal [11], [12] and Porwal and Aouf [13] etc. presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Duren [6], Ponnusamy and Rasila [10] and references therein for basic results on the subjects.

For $f=h+\overline{g}$ given by (1.1), Jahangiri et al. [8] defined the modified Salagean operator of f as

$$D^m f(z)=D^m h(z)+(-1)^m\overline{D^m g(z)}\ , \qquad \left(m\in N_0, N_0=N\cup\{0\}\right) \tag{1.2}$$

where

$$D^m h(z)=z+\sum_{k=2}^{\infty}k^m a_k z^k \ \text{ and } \ D^m g(z)=\sum_{k=1}^{\infty}k^m b_k z^k\ ,$$

where D^n stands for the differential operator introduced by Salgean [14].

Now for $0\le\alpha<1,\ n\in N_0$ and $z\in U$, suppose that $S_H(n,\alpha)$ denote the family of harmonic univalent functions f of the form (1.1) such that

$$\operatorname{Re}\left\{\frac{D^n h(z)+D^n g(z)}{z}\right\}>\alpha\,, \tag{1.3}$$

where $D^n f$ is defined by (1.2).

Further let the subclass $\overline{S}_H(n,\alpha)$ consisting of harmonic functions $f=h+\overline{g}$ in $\overline{S}_H(n,\alpha)$ so that h and g are of the form

$$h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k \ \text{ and } \ g(z)=-\sum_{k=1}^{\infty}|b_k|z^k\,. \tag{1.4}$$

We note that $S_H(1,\alpha)=HP(\alpha)$ and $\overline{S_H}(1,\alpha)=HP^*(\alpha)$ studied by Yalcin et al. in [17].

Clearly if $0\le\alpha_1\le\alpha_2<1$, then $S_H(n,\alpha_2)=S_H(n,\alpha_1)$.

In the present paper, results involving the coefficient inequalities, distortion bounds are obtained.

2. Main Results

We begin by proving some sharp coefficient inequalities contained in the following theorem.

Theorem 2.1. Let the function $f=h+\overline{g}$ be such that *h* and *g* are given by (1.1). Furthermore

$$\sum_{k=2}^{\infty}k^n|a_k|+\sum_{k=1}^{\infty}k^n|b_k|\le 1-\alpha, \tag{2.1}$$

where $0\le\alpha<1$ and $n\in N_0$. Then f is harmonic univalent, sense-preserving in U and $f\in S_H(n,\alpha)$.

Proof. . If $z_1\ne z_2$, then,

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{z_1 - z_2 + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|}$$

$$\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{k^n}{1-\alpha} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{k^n}{1-\alpha} |a_k|}$$

$$\geq 0.$$

Hence f is univalent in U.

F is sense-preserving in U. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1}$$

$$> 1 - \sum_{k=2}^{\infty} k |a_k|$$

$$\geq 1 - \sum_{k=2}^{\infty} \frac{k^n}{1-\alpha} |a_k|$$

$$\geq \sum_{k=1}^{\infty} \frac{k^n}{1-\alpha} |b_k|$$

$$\geq \sum_{k=1}^{\infty} k |b_k| |z|^{k-1}$$

$$\geq |g'(z)|.$$

Now we show that $f \in S_H(n, \alpha)$. Using the fact that $\operatorname{Re} \omega > \alpha$ if and only if $|1 - \alpha + \omega| > |1 + \alpha - \omega|$, it suffices to show that

$$\left| (1-\alpha) + \frac{D^n h(z) + D^n g(z)}{z} \right| - \left| (1+\alpha) - \frac{D^n h(z) + D^n g(z)}{z} \right| > 0. \qquad (2.2)$$

Substituting for $D^n h(z)$ and $D^n g(z)$ in L.H.S. of (2.2), we have

$$= \left| (2-\alpha) + \sum_{k=2}^{\infty} k^n a_k z^{k-1} + \sum_{k=1}^{\infty} k^n b_k z^{k-1} \right| - \left| \alpha - \sum_{k=2}^{\infty} k^n a_k z^{k-1} - \sum_{k=1}^{\infty} k^n b_k z^{k-1} \right|$$

$$\geq 2(1-\alpha) \left\{ 1 - \sum_{k=2}^{\infty} \frac{k^n}{1-\alpha} |a_k| |z|^{k-1} - \sum_{k=1}^{\infty} \frac{k^n}{1-\alpha} |b_k| |z|^{k-1} \right\}$$

$$> 2(1-\alpha)\left\{1-\sum_{k=2}^{\infty}\frac{k^n}{1-\alpha}|a_k|-\sum_{k=1}^{\infty}\frac{k^n}{1-\alpha}|b_k|\right\}\geq 0.$$

The harmonic mappings

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{k^n}x_k z^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{k^n}\overline{y_k z^k},$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that coefficient bound given by (2.1) is sharp.

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f=h+\overline{g}$, where h and g are of the form (1.4).

Theorem2.2. Let $f=h+\overline{g}$ be given by (1.4). Then $f\in\overline{S}_H(n,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}\frac{k^n}{1-\alpha}|a_k|+\sum_{k=1}^{\infty}\frac{k^n}{1-\alpha}|b_k|\leq 1. \tag{2.3}$$

where $0\leq\alpha<1$ and $n\in N_0$.

Proof. The if part follows from Theorem 2.1. For the only if part, we show that $f\in\overline{S}_H(n,\alpha)$, if the condition (2.3) holds. We notice that the condition

$$\operatorname{Re}\left\{\frac{D^n h(z)+D^n g(z)}{z}\right\}>\alpha$$

is equivalent to

$$\operatorname{Re}\left\{1-\sum_{k=2}^{\infty}k^n|a_k|z^{k-1}-\sum_{k=1}^{\infty}k^n|b_k|z^{k-1}\right\}>\alpha.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$1-\sum_{k=2}^{\infty}k^n|a_k|-\sum_{k=1}^{\infty}k^n|b_k|\geq\alpha$$

which is precisely the assertion (2.3).

Next, we determine the extreme points of closed convex hulls of $\overline{S}_H(n,\alpha)$ denoted by clco $\overline{S}_H(n,\alpha)$.

Theorem 2.3. Let f be given by (1.4). Then $f\in\overline{S}_H(n,\alpha)$ if and only if

$$f(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_k(z)\right),$$

where $h_1(z)=z$,

$$h_k(z)=z-\frac{1-\alpha}{k^n}z^k,(k=2,3,4....),\ g_k(z)=z-\frac{1-\alpha}{k^n}\overline{z}^k,(k=1,2,3....)\quad ,x_k\geq 0,\ y_k\geq 0,\ \sum_{k=1}^{\infty}(x_k+y_k)=1.$$ In particular the extreme points of $\overline{S}_H(n,\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

The following theorem gives the bounds for functions in $\overline{S}_H(n,\alpha)$ which yields a covering result for this class.

Theorem 2.4. Let $f \in \overline{S}_H(n,\alpha)$. Then for $|z| = r < 1$ we have

$$|f(z)| \le (1+|b_1|)r + \frac{1}{2^n}(1-|b_1|-\beta)r^2, \ |z| = r < 1$$

and

$$|f(z)| \ge (1-|b_1|)r - \frac{1}{2^n}(1-|b_1|-\beta)r^2, \ |z| = r < 1.$$

Proof. The proofs of the above Theorems 2.3 and 2.4 are analogues to the corresponding similar theorems proved in [17] and therefore we omit the details involved.

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