DESIGNING OF A NOVEL MATHEMATICAL MODEL FOR THE ANALYSIS OF AIDS TRANSMISSION CAUSED BY CO- INFECTION OF HIV WITH TB

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ABSTRACT

This manuscript describes a combined SIS-SIRS model with migration for the spread of AIDS caused in co- infection involving TB and HIV is proposed. In modeling the problem, it is assumed that an HIV infected individual's interacts with TB infected persons the former to have HIV & TB. It is assumed further that a fraction of this HIV class becomes AIDS patients. It is also considered that a certain fraction of TB class form a reservoir class, a part of which may recover and becomes susceptibles again. The model is analysed by using stability theory of differential equation. It is shown that AIDS epidemic spread faster due to co- infection of HIV with TB.

INTRODUCTION

From the first model it contains a new class of patients who were suffering from both T.B. &AIDS .So it has different calculations are included.

The increasing incidence of various infectious diseases such as gonorrhea,T.B.,HIV, AIDS,etc. is causing major public helth problems in the most developed and developing countries ,[1] .In general the spread of such diseases in human population depends upon various factors such as the numbers of infectives ,susceptibles,modesoftrasmission(social,ecological,andgeographicalconditions)[2].Anaccount of modeling and study of epidemic diseases can be found in lectures notes by [3]and books by [4]Many infectious diseases are spread by direct contact between susceptibles and infectives,while some others are spread indirectly by carriersbacteriaorvectors [5]Tuberculosis (TB) is an infectious disease caused by a bacteria called Mycobacterium tuberculosis and affects lung and other parts of the body. In developing countries pulmonary tuberculosis (i.e., T.B.) in the lung accounts more than seventy percent of cases in the world. T.B. , which had been controlled to a significant extent earlier, has re-emerged in recent years as one of the leading causes of death . Nearly three million people die every years due to T.B. Any one can become infected with T. B.bacteria but people with H.I.V.are at greater risk of getting infected with this disease. T.B.is transmitted mainly by droplet nuclei when the person suffering from active T.B. breathes, sneezes or coughs in the air[6] AIDS , the acquired immuno-deficiency syndrome is a fatal diease caused by a retrovirus known as the human immuno deficiency virus (HIV) which although attacks many blood cells it causes havoc on the T-cell in the blood by destroying and decreasing their number leading to decline in body's immunity to fight infection . The term AIDS refers to only the last stage of the HIV infection after which death occurs . The AIDS epidemic is sweeping the world both developed and developing nations as HIV has infected millions of people all over the world without any restriction of nationalty , colour , caste or reliogion , etc. HIV infection in adults (75-80^%) has been transmitted to one partner through unprotected sexual inter course when the other partner is infected with HIV . Mother to child transmission (vertical infection accounts more than 90^% of global infection to infants and children . Apart from these , other possible source of infection are blood transfusion, intravenous drug addiction etc . In general sexually transmitted disease(STD) such as gonorrhea induce little or no acquired immunity upon recovery but in case of HIV immunity decreases[4]. The transmission dynamics of HIV infection is affected by various factors such as latent period , the number of infected/infectious person in the population, having AIDS , type of sexual activity , type of mixing (homogeneous or heterogeneous, various high risk groups , etc. It is noted that in case of HIV infected persons immunity decreases and they can early infected by other diseases such as TB forming a class HIV co infected with TB. Some efforts have been to understand the transmission of co infection of HIV with TB infections .mathematical models ,Taking into account fixed values for the transmission coefficients ,these models dealt with the dynamic simulations and the equilibrium point analysis .However ,if the equilibrium point analyzes could be achieved for all finite the transmission coefficients ,then it would provides us different scenarios of the interaction between HIV and TB infections.

In this paper a model for the transmission of co infection HIV with TB infections and examine some of the epidemiological implications.

A COMBINED SIS-SIRS MODEL WITH CONSTANT IMMIGRATION

In this section a combined SIS-SIRS model with constant immigration of human population is proposed and analyzed.

In proposing the model, the following assumptions are made.

(1).All HIV infected individuals are considered susceptibles to TB infection and, at the same manner, all TB infected individuals are considered susceptibles to HIV.

(2).Due to loss of immunity HIV infected persons get infected with TB after interaction.

(3).There is a separate class of Pearson infected both with HIV and TB.The HIV class also includes persons having co infection.

Based on the above assumptions we propose a mathematical model to assess the spread of co infection HIV with TB infections .It is assumed that the population density N(t) is divided into five classes , the susceptible density X(t),the density of population infected only with TB(Y_T),the density of population infected with HIV(Y_H)a fraction of which may be also infected with TB ,the density of AIDS class(Y_a) and the density of co infected class. The models is proposed as follows,

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + \nu_1 Y_T$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - \nu_1 Y_T$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} - \frac{\beta_4 Y_T Y_H}{N} - dY_H - \mu Y_H$$

$$\frac{dY_{TH}}{dt} = \frac{\beta_3 Y_T Y_H}{N} + \frac{\beta_4 Y_T Y_H}{N} - dY_{TH} - \mu_1 Y_{TH}$$

$$\frac{dY_a}{dt} = \mu Y_H + \mu_1 Y_{TH} - dY_a - \alpha Y_a$$

$$N = X + Y_T + Y_H + Y_{TH} + Y_a$$

Where

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

in this model , β_1 and β_2 are the contact-rates by which susceptibles decrease following contact with persons having of TB and AIDS and β_3, β_4 be the co-infection constant . The clinical AIDS developed at the rate μ per year. ν_1 is the recovery rate at which TB infectives becomes susceptibles again ,d is the natural death rate ,α be death rate due to AIDS. μ_1 be the rate at which co infected class becomes AIDS susceptibles.

3.EQUILIBRIUM ANALYSIS;

To analyze the model ,we consider the following reduced system. Since $N = X + Y_T + Y_H + Y_{TH} + Y_a$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_{TH} - Y_a)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + \nu_1) Y_T$$
$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_H - Y_{TH} - Y_a)}{N} - \frac{\beta_4 Y_T Y_H}{N} - (d + \mu) Y_H$$
$$\frac{dY_{TH}}{dt} = \frac{\beta_3 Y_T Y_H}{N} + \frac{\beta_4 Y_T Y_H}{N} - dY_{TH} - \mu_1 Y_{TH}$$
$$\frac{dY_a}{dt} = \mu Y_H + \mu_1 Y_{TH} - (d + \alpha) Y_a$$
$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

……………………………………………….(2)

The results of the equilibrium analysis are given in the following theorems;
THEOREM-There exists following four equilibria of the model discussed before.

(i) $E_0 = \left(0,0,0,0,\frac{A}{d}\right)$

(ii) $E_1 = (Y_T, 0,0,0, N)$ Which exists if $\beta_1 \rangle d + \nu_1$

where $\bar{Y}_T = \frac{A(\beta_1 - d - \nu_1)}{d\beta_1}, \bar{N} = \frac{A}{d}$

(iii) $E_2 = (0, \hat{Y}_H, \hat{Y}_a, 0, N)$, Which exists if $\beta_2 \rangle d + \mu$

where

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$
$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$
$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

(iv)The general case $E_3 = (Y_T^*, Y_H^*, Y_{TH}^*, Y_a^*, N^*)$ is discussed below as follows;

$$Y_T^* = \frac{a_3 a_1 - a_1 b_2}{a_2 a_3 - a_1 a_4}, Y_H^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3}$$

Where the constants are given by

$$a_1 = -\beta_1 - \frac{\beta_1\beta_4(\beta_1 - d - \nu_1)}{\beta_1(d+\mu) - \beta_2(d+\nu_1)} - \frac{\beta_1^2\beta_4 d(d+\alpha)}{\alpha\mu_1[\beta_1(d+\mu) - \beta_2(d+\nu_1)]} - \frac{\beta_1^2\beta_4 d}{\alpha[\beta_1(d+\mu) - \beta_2(d+\nu_1)]}$$

$$a_2 = -\beta_1 - \beta_3 + \frac{\mu\beta_1}{\mu_1} + \frac{\beta_2\beta_3(\beta_1 - d - \nu_1)}{[\beta_1(d+\mu) - \beta_2(d+\nu_1)]} + \frac{d\beta_1\beta_2\beta_3(d+\alpha)}{\alpha\mu_1[\beta_1(d+\alpha) - \beta_2(d+\nu_1)]} + \frac{d\beta_1\beta_2\beta_3}{\alpha[\beta_1(d+\mu) - \beta_2(d+\nu_1)]}$$

$$a_3 = -\beta_2 - \beta_4 - \frac{\beta_1\beta_4(\beta_2 - d - \mu)}{[\beta_1(d+\mu) - \beta_2(d+\nu_1)]} - \frac{\beta_1\beta_2\beta_4 d(d+\alpha)}{\alpha\mu_1[\beta_1(d+\mu) - \beta_2(d+\nu_1)]} - \frac{d\beta_1\beta_2\beta_4}{\alpha[\beta_1(d+\mu) - \beta_2(d+\nu_1)]}$$

$$a_4 = \frac{\beta_2\beta_3(\beta_2 - d - \mu)}{[\beta_1(d+\mu) - \beta_2(d+\nu_1)]} - \beta_2 + \frac{\mu\beta_2}{\mu_1} + \frac{\beta_2^2 d(d+\alpha)\beta_3}{\alpha\mu_1[\beta_1(d+\mu) - \beta_2(d+\nu_1)]} + \frac{d\beta_2^2\beta_3}{\alpha[\beta_1(d+\mu) - \beta_2(d+\nu_1)]}$$

$$b_1 = \frac{A\beta_1}{\alpha}[1 + \frac{d+\alpha}{\mu_1}] \; , \; b_2 = \frac{A\beta_2}{\alpha}[1 + \frac{d+\alpha}{\mu_1}]$$

it will be +ive if $b_1 a_4 \langle a_2 b_2, a_1 a_4 \rangle a_2 a_3, a_1 b_2 \rangle a_3 b_1, a_2 \rangle 0, a_4 \rangle 0$ i.e.

$$\frac{\beta_2}{\beta_1}(d+\nu_1) - (d+\mu) \rangle 0$$

$$(\beta_1 - d - \nu_1) - \frac{\beta_1 A(\beta_2 - d - \mu)}{d\beta_2} - \frac{\beta_3 A(d+\alpha)(\beta_2 - d - \mu)}{d\beta_2(d+\alpha+\mu)} \rangle 0$$

(FOR PROOF SEE APPENDIX—A)

4 STABILITY ANALYSIS;-

Now we study the local stability of equilibria E_0, E_1, E_2 *and* E_3 and the non-linear stability of non trivial equilibrium E_3. The results are stated in the form of the following theorems.

THEOREM - 1

The equilibrium E_0, E_1, E_2 is unstable provided the following conditions are satisfied.

(i) $\beta_1 \rangle d + \nu_1 , \; \beta_2 \rangle d + \mu$

(ii) $\frac{\beta_2(d+\nu_1)}{\beta_1} - (d+\mu) \rangle 0$

(iii) $(\beta_1 - d - \nu_1) - \frac{\beta_1 A(\beta_2 - d - \mu)}{d\beta_2} - \frac{\beta_3 A(d+\alpha)(\beta_2 - d - \mu)}{d\beta_2(d+\alpha+\mu)} \rangle 0$

(FOR PROOF SEE APPENDIX –B)

THEOREM -2

The equilibrium $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$ is locally asymptotically stable provided that following conditions are satisfied.

(i) $[c_1(\beta_1 + \beta_3)\frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3)\frac{Y_H^*}{N^*}]^2 \langle \frac{c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{4N^{*2}}$

(ii) $c_1 \beta_1 Y_T^* \langle \frac{c_3 N^*(d+\alpha)}{4}$

(iii) $\left(\frac{c_1\beta_1 Y_T^*}{N^*} - c_4 \nu_2\right)^2 \langle \frac{c_1 c_4(\theta + d)\beta_1 Y_T^*}{N^*}$

(iv) $c_1 Y_T^*(\beta_1 - d - \nu_1 - \nu_2)^2 \langle \frac{c_5 \beta_1 N^* d}{4}$

(v) $[\frac{c_2\beta_2 Y_H^*}{N^*} - \mu c_3]^2 \langle \frac{c_2 c_3(d+\alpha)\beta_2 Y_H^*}{4N^*}$

(vi) $c_2\beta_2 Y_H^* \langle \frac{c_4(d+\theta)N^*}{4}$

(vii) $c_2 Y_H^*(\beta_2 - d - \mu)^2 \langle \frac{c_5 d \beta_2 N^*}{4}$

(viii) $c_3 c_4 \rangle 0$

(ix) $c_5 \alpha^2 \langle \frac{c_3(d+\alpha)}{4}$

(x) $c_4 c_5 \rangle 0$

[FOR PROOF SEE APPENDIX –C]

THEOREM-3

The equilibrium $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$ is a non linear stable provided the following conditions are satisfied

$(i)(\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 \langle \frac{\beta_1\beta_2 m_1}{4}$

(ii) $\left(\frac{\beta_2 m_1}{N^*} - \mu m_2\right)^2 \langle \frac{\beta_1 m_2(d+\alpha)}{4N^*}$

(iii) $(4\beta_1 + \beta_3)^2 \langle \frac{d m_3 \beta_1 N^*}{4}$

(iv) $\left(\frac{-\beta_1}{N^*} + \nu_2 m_4\right)^2 \langle \frac{m_4\beta_1(d+\theta)}{4N^*}$

(v) $\left(\frac{-\beta_2 m_1}{N^*} + \mu m_2\right)^2 \langle \frac{m_1 m_2(d+\alpha)\beta_2}{4N^*}$

(vi) $(4\beta_2 - \beta_3)^2 \frac{m_1}{N^*} \langle \frac{d m_3 \beta_2}{4}$

(vii) $\left(\frac{\beta_2^2 m_1}{N^*}\right) \langle \frac{m_4(d+\theta)\beta_2}{4}$

(viii) $m_3\alpha^2 \langle \frac{m_2 d(d+\alpha)}{4}$

(ix) $m_2 m_4, m_3 m_4 \rangle 0$

to prove the above theorem we need the following leema,

the region of attraction is given by the following set.

$$0 \le Y_T \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d\beta_1}\ ,\ 0 \le Y_H \le \frac{A(\beta_2 - d - \mu)}{d\beta_2}, 0 \le Y_a \le \frac{\mu A((\beta_2 - d - \mu)}{d\beta_2(d+\alpha)}$$

$$0 \le Z \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d\beta_1(d+\theta)}\ ,\ 0 \le N \le \frac{A}{d}$$

[FOR PROOF SEE APPENDIX-D]

A PARTICULAR CASE

SIS MODEL WITH CONSTANT IMMIGRATION

If in the model if we put $\nu_2 = 0$, θ=0 it becomes sis model .

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + \nu_1 Y_T$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - \nu_1 Y_T$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a$$

where

$$N = X + Y_T + Y_H + Y_a$$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

all the symbols have their same meaning as in the previous model.

EQUILIBRIUM ANALYSIS

In this case there exists four equilibria of the model.

1; $E_0 = \left(0,0,0,,\frac{A}{d}\right)$

2; $E_1 = (Y_T, 0,0,, N)$ which exists if $\beta_1 \rangle d + \nu_1$

where

$$Y_T = \frac{A(\beta_1 - d - \nu_1)}{d\beta_1}\quad , N = \frac{A}{d}$$

3; $E_2 = (0, \hat{Y}_H, \hat{Y}_a, N)$, Which exists if $\beta_2 \rangle d + \mu$

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

where

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

4; $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$, This is defined in three parts ;

(a) $E_{30}(Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, N_0^*)$ Where

$Y_{TO}^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4}$, $Y_{H0}^* = \frac{b_2}{a_4}$, $Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}$, $N_0^* = \frac{A}{d} - \frac{\alpha}{d}\frac{\mu b_2}{a_4(\alpha + d)}$

it exists only when

$$\beta_2 \rangle d + \mu$$

$$\left[\{\beta_1(d+\mu) - \beta_2(d+\nu_1)\}(1 + \frac{\mu}{\alpha + d}) - \beta_3(\beta_2 - d - \mu) \rangle 0 \right]$$

$E_{31}(Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, N_1^*)$ Where

$Y_{T1}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3}$, $Y_{H1}^* = \frac{a_1 b_2 + a_3 b_3}{a_1 a_4 + a_2 a_3}$, $Y_a = \frac{\mu Y_H^*}{d + \alpha}$

$N_1^* = \frac{A}{d} - \frac{\alpha \mu Y_{H1}^*}{d(d+\alpha)}$, *it exists providedthe following conditions are satisfied.*

$$[\beta_1(d+\mu) - \beta_2(d+\nu_1)](1 + \frac{\mu}{d+\alpha}) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

(b) $\beta_1\beta_2\left[\frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d+\alpha}\right] - \beta_3(\beta_2 - d - \mu)\frac{\alpha\mu}{d} - \mu(d+\alpha)\beta_2 \rangle 0$

© $E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, Z_2^*, N_2^*)$,(*THE GENERAL CASE*)

Where

$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}$, $Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}$, $Y_{a2}^* = \frac{\mu Y_{H2}^*}{d+\alpha}$

$N_2^* = \frac{A}{d} - \frac{\alpha \mu Y_{H2}^*}{d(d+\alpha)}$, *it exists provided that*

$a_4 b_1 \rangle a_2 b_2$, *and* $a_1 b_2 \rangle a_3 b_1$,*i.e.*

$$[\beta_1(d+\mu) - \beta_2(d+\nu_1)](1 + \frac{\mu}{d+\alpha}) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - \nu_1 - \nu_2)}{\beta_1} - (d+\mu) \rangle 0 ,$$

where

$$a_1 = \beta_1 , \; a_2 = \frac{\alpha\mu(\beta_1 - d - \nu_1)}{d(d+\alpha)} + \beta_1 + \frac{\beta_1\mu}{d+\alpha} + \beta_3 , \; a_3 = \beta_2 - \beta_3$$

$$a_4 = \frac{\alpha\mu(\beta_2 - d - \mu)}{d(d+\alpha)} + \beta_2 + \frac{\mu\beta_2}{d+\alpha} , \; b_1 = \frac{A(\beta_1 - d - \nu_1)}{d} , \; b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

(FOR PROOF SEE APPENDIX—A)

STABILITY ANALYSIS

now we study the local stability of equilibria E_0, E_1, E_2 *and* E_3 and the non-linear stability of non trivial equilibrium E_3. The results are stated in the form of the following theorems .

THEOREM - 1;

The equilibrium E_0, E_1, E_2 is unstable provided the following conditions are satisfied .

(i) $\beta_1 \rangle d + \nu_1$, $\beta_2 \rangle d + \mu$

(ii) $\frac{\beta_2 d}{\beta_1}+\frac{\beta_3(\beta_1-d-\nu_1)}{\beta_1}-(d+\mu)\ \rangle 0$

(iii) $[\beta_1(d+\mu)-\beta_2(d+\nu_1)](1+\frac{\mu}{d+\alpha})-\beta_3(\beta_2-d-\mu)\ \rangle 0$

(FOR PROOF SEE APPENDIX –B)

THEOREM -2

The equilibrium $E_3=(Y_T^*,Y_H^*,Y_a^*,N^*)$ is locally asymptotically stable provided that following conditions are satisfied.

(i) $[c_1(\beta_1+\beta_3)\frac{Y_T^*}{N^*}+c_2(\beta_2-\beta_3)\frac{Y_H^*}{N^*}]^2\ \langle\ \frac{4c_1c_2\beta_1\beta_2Y_T^*Y_H^*}{9N^{*2}}$

(ii) $c_1\beta_1Y_T^*\ \langle\ \frac{4c_3N^*(d+\alpha)}{9}$

(iii) $c_1Y_T^*(\beta_1-d-\nu_1)^2\ \langle\ \frac{4c_4\beta_1N^*d}{9}$

(iv) $[\frac{c_2\beta_2Y_H^*}{N^*}-\mu c_3]^2\ \langle\ \frac{4c_2c_3(d+\alpha)\beta_2Y_H^*}{9N^*}$

(v) $c_2Y_H^*(\beta_2-d-\mu)^2\ \langle\ \frac{4c_4d\beta_2N^*}{9}$

(vi) $c_4\alpha^2\ \langle\ \frac{4c_3(d+\alpha)}{9}$

[FOR PROOF SEE APPENDIX-C]

THEOREM-3

The equilibrium $E_3=(Y_T^*,Y_H^*,Y_a^*,N^*)$ is a non linearly stable provided the following conditions are satisfied

(i) $(\beta_1+\beta_3+\beta_2m_1-\beta_3m_1)^2\ \langle\ \frac{4\beta_1\beta_2m_1}{9}$

(ii) $\beta_1\langle\frac{4m_2(d+\alpha)N^*}{9}$

(iii) $(3\beta_1+\beta_3)^2\ \langle\ \frac{4dm_3\beta_1N^*}{9}$

(iv) $\left(\frac{-\beta_2m_1}{N^*}+\mu m_2\right)^2\ \langle\ \frac{4m_1m_2(d+\alpha)\beta_2}{9N^*}$

(v) $(\beta_2m_1+\beta_2m_2+\beta_3m_1)^2\langle\frac{4\beta_2m_1m_3dN^*}{9}$

(vi) $m_3\alpha^2\ \langle\ \frac{4m_2d(d+\alpha)}{9}$

To prove the above theorem we need the following lemma.

The region of attraction is

$$0\le Y_T\le\frac{A(\beta_1-d-\nu_1)}{d\beta_1},\ 0\le Y_H\le\frac{A(\beta_2-d-\mu)}{d\beta_2},0\le Y_a\le\frac{\mu A((\beta_2-d-\mu)}{d\beta_2(d+\alpha)},\ 0\le N\le\frac{A}{d}$$

[FOR PROOF SEE APPENDIX-D]

CONCLUSION

In this paper , a non linear model is proposed and analyzed to study the co- infection of TB and AIDS.It is shown that the spread of AIDS increases due to co-infection of TB and AIDS and the disease becomes endemic. It is also found that due to migration of HIV infection the spread of AIDS increases further.

APPENDIX-A

The given model is

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + \nu_1 Y_T + \theta Z \quad \text{..........................} (A_1)$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - \nu_1 Y_T - \nu_2 Y_T \quad \text{...................................} (A_2)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \quad \text{...................................} (A_3)$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \quad \text{..} (A_4)$$

$$\frac{dZ}{dt} = \nu_2 Y_T - dZ - \theta Z \quad \text{...} (A_5)$$

Where $N = X + Y_T + Y_H + Y_a + Z$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad \text{...} (A_6)$$

the reduced system is given below,

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + \nu_1 + \nu_2) Y_T \text{......................} (A_7)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_a - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu) Y_H \text{..} (A_8)$$

$$\frac{dY_a}{dt} = \mu Y_H - (d + \alpha) Y_a \text{..} (A_9)$$

$$\frac{dZ}{dt} = \nu_2 Y_T - dZ - \theta Z \text{..} (A_{10})$$

$$\frac{dN}{dt} = A - dN - \alpha Y_a \text{...} (A_{11})$$

EXISTENCE OF $E_0(0,0,0,0,\frac{A}{d})$

It is obvious from the model $(A_7 \text{..........} A_{11})$,we can find if $Y_T = Y_H = Y_a = 0$

So $N = \frac{A}{d}$

This is diease free equilibrium.

EXISTENCE OF $E_1(\bar{Y}_T, 0, 0, \bar{Z}, \bar{N})$

Since $Y_H = 0 \Rightarrow Y_a = 0$ and $Y_T \neq 0$

From (A_{11}) we get $\bar{N} = \frac{A}{d}$, From (A_{10}), we get $Z = \frac{\nu_2 Y_T}{\theta + d}$

from (A_7) we get

$$\beta_1 Y_T \left(\bar{N} - \bar{Y}_T - \bar{Z}\right) - \left(d + \nu_1 + \nu_2\right)\bar{Y}_T \bar{N} = 0$$

Since $\bar{Y}_T \neq 0$

So on solving , we get

$$\bar{Y}_T = \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d\beta_1 \nu}, \bar{Z} = \frac{\nu_2 Y_T}{d + \theta}, \bar{N} = \frac{A}{d}$$

Which exists if $\beta_1 \rangle d + \nu_1 + \nu_2$

EXISTENCE OF $E_2(0, \hat{Y}_H, \hat{Y}_a, 0, \hat{N})$

Since $\hat{Y}_T = 0 \Rightarrow \hat{Z} = 0, And\ \hat{Y}_H, \hat{Y}_a \neq 0$

So ,on solving A_8 , A_9 , A_{11} We get,

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

, Which exists if $\beta_2 \rangle d + \mu$

EXISTENCE OF $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$

From(A_9), we get $Y_a^* = \frac{\mu Y^*{}_H}{d + \alpha}$, from (A_{10}) We get $Z^* = \frac{\nu_2 Y_T}{d + \theta}$

From (A_{11}) $N^* = \frac{A}{d} - \frac{\alpha\mu Y_H^*}{d(d + \alpha)}$

So A_7 and A_8 Becomes

$$\beta_1 \nu Y^*{}_T + Y_H^* \left[\frac{\alpha\mu(\beta_1 - d - \nu_1 - \nu_2)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1 \mu}{d + \alpha} + \beta_3\right] = \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A_{12})$$

$$Y_T^*(\beta_2 \nu - \beta_3) + Y_H^* \left[\frac{\alpha\mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha}\right] = \frac{A(\beta_2 - d - \mu)}{d} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A_{13})$$

where

$$a_1 = \nu\beta_1, \ a_2 = \frac{\alpha\mu(\beta_1 - d - \nu_1 - \nu_2)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1 \mu}{d + \alpha} + \beta_3, \ a_3 = \beta_2 \nu - \beta_3$$

$$a_4 = \frac{\alpha\mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha}, \ b_1 = \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d}, \ b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

$$\nu = 1 + \frac{\nu_2}{d + \theta}$$

so A_{12} And A_{13} Becomes

$$a_1 Y_T^* + a_2 Y_H^* = b_1, \ a_3 Y_T^* + a_4 Y_H^* = b_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A_{14})$$

now here arise the three cases –

(1);if $a_3 = 0 \Rightarrow \beta_2 \nu = \beta_3$ *so the equilibrium* $E_{30}(Y^*_{TO}, Y^*_{HO}, Y^*_{a0}, Z^*_0, N^*_0)$

so

$$Y^*_{TO} = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4} \ , \ Y^*_{H0} = \frac{b_2}{a_4}, \ Y^*_{a0} = \frac{\mu b_2}{a_4(\alpha + d)}$$

$$Z^*_0 = \frac{\nu_2(b_1 a_4 - a_2 b_2)}{a_1 a_4}, \ N^*_0 = \frac{A}{d} - \frac{\alpha}{d}\frac{\mu b_2}{a_4(\alpha + d)}$$

it exists only when

$$\beta_2 \rangle d + \mu$$

$$\left[\{\nu\beta_1(d+\mu) - \beta_2(d + \nu_1 + \nu_2)\}(1 + \frac{\mu}{\alpha + d}) - \beta_3(\beta_2 - d - \mu) \rangle 0\right]$$

(2)if < 0 so The Equilibrium b) $E_{31}(Y^*_{T1}, Y^*_{H1}, Y^*_{a1}, Z^*_1, N^*_1)$ *Where*

$$Y^*_{T1} = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3} \ , Y^*_{H1} = \frac{a_1 b_2 + a_3 b_3}{a_1 a_4 + a_2 a_3} \ , , \ Z^*_1 = \frac{\nu_2 Y^*_{T1}}{d + \theta}$$

$N^*_1 = \frac{A}{d} - \frac{\alpha\mu Y^*_{H1}}{d(d+\alpha)}$, *it exists providedthe following conditions are satisfied.*

$$[\nu\beta_1(d+\mu) - \beta_2(d + \nu_1 + \nu_2)](1 + \frac{\mu}{d+\alpha}) - \beta_3(\beta_2 - d - \mu)\rangle 0$$

$$\nu\beta_1\beta_2\left[\frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d+\alpha}\right] - \beta_3(\beta_2 - d - \mu)\frac{\alpha\mu}{d} - \mu(d+\alpha)\beta_2 \ \rangle 0$$

(3) if $a_3 > 0$ *soThe Equilibrium* © $E_{32}(Y^*_{T2}, Y^*_{H2}, Y^*_{a2}, Z^*_2, N^*_2)$,(*THE GENERAL CASE*)

Where

$$Y^*_{T2} = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y^*_{H2} = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y^*_{a2} = \frac{\mu Y^*_{H2}}{d + \alpha}$$

$Z^*_2 = \frac{\nu_2 Y^*_{T1}}{d+\theta}, \ N^*_2 = \frac{A}{d} - \frac{\alpha\mu Y^*_{H2}}{d(d+\alpha)}$, *it exists provided that*

$a_4 b_1 \rangle \ a_2 b_2$, *and* $a_1 b_2 \rangle a_3 b_1$,*i.e.*

$$[\nu\beta_1(d+\mu) - \beta_2(d + \nu_1 + \nu_2)](1 + \frac{\mu}{d+\alpha}) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - \nu_1 - \nu_2)}{\beta_1} - (d+\mu)\nu \rangle 0 \ ,$$

APPENDIX-B

Using the model $(A_7 ....A_{11})$, The general jacobian matrix can be written as follows;

$$M = \begin{array}{ccccc} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ 0 & \mu & -(\alpha+d) & 0 & 0 \\ \nu_2 & 0 & 0 & -(\theta+d) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{array}$$

where

$$m_{11} = \frac{\beta_1(N-Y_T-Y_H-Y_a-Z)}{N} - \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_H}{N} - d - \nu_1 - \nu_2 \; m_{12} = \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_T}{N}, m_{13} = \frac{-\beta_1 Y_T}{N}, m_{14} = \frac{-\beta_1 Y_T}{N}$$

$$m_{15} = \frac{\beta_1 Y_T}{N} - \frac{\beta_1 Y_T(N-Y_T-Y_H-Y_a-Z)}{N^2} + \frac{\beta_3 Y_T Y_H}{N^2}$$

$$m_{21} = \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_H}{N}, m_{22} = \frac{\beta_2(N-Y_T-Y_H-Y_a-Z)}{N} - \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_T}{N} - (d+\nu)$$

$$m_{23} = \frac{-\beta_2 Y_H}{N}, \quad m_{24} = \frac{-\beta_2 Y_H}{N}$$

$$m_{25} = \frac{\beta_2 Y_H}{N} - \frac{\beta_2 Y_H(N-Y_T-Y_H-Y_a-Z)}{N^2} - \frac{\beta_3 Y_T Y_H}{N^2}$$

PROOF OF THEOREM-1 stability of $E_0(0,0,0,0,\frac{A}{d})$

The jacobian matrix for E_0 is given by

$$M(E_0) = \begin{array}{ccccc} \beta_1-d-\nu_1-\nu_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_2-d-\mu & 0 & 0 & 0 \\ 0 & \mu & -(d+\alpha) & 0 & 0 \\ \nu_2 & 0 & 0 & -(d+\theta) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{array}$$

the eigen values are $\beta_1 - d - \nu_1 - \nu_2$, $\beta_2 - d - \mu$, $-(d+\alpha)$, $-(d+\theta)$,-d

since $\beta_1 \rangle d+\nu_1+\nu_2$, $\beta_2 \rangle d+\mu$ so the equilibrium E_0 is unstable.

Stability for $E_1(\overline{Y}_T,0,0,\overline{Z},\overline{N})$

The jacobian matrix for E_1 is given as

$$M(E_1) = \begin{array}{ccccc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & \mu & -(\alpha+d) & 0 & 0 \\ \nu_2 & 0 & 0 & -(d+\theta) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{array}$$

where $b_{11} = -(\beta_1 - d - \nu_1 - \nu_2)$, $b_{12} = -(\nu\beta_1 + \beta_3)(\beta_1 - d - \nu_1 - \nu_2)$, $b_{13} = b_{14} = -(\beta_1 - d - \nu_1 - \nu_2)$

$$b_{15} = \frac{(\beta_1 - d - \nu_1 - \nu_2)}{\beta_1} d^2$$

$$b_{22} = \frac{\beta_2}{\beta_1} d + \frac{\beta_3}{\beta_1} (\beta_1 - d - \nu_1 - \nu_2) - \nu(d + \mu)$$

Since eigen value are , $b_{11}, b_{22}, -(\alpha + d), -(\theta + d), -d$

Since $b_{22} > 0$ so E_1 is unstable

stability of $E_2(0, \hat{Y}_H, \hat{Y}_a, 0, \hat{N})$,

The jacobian for $E_2 is$

$$M(E_2) = \begin{matrix} \hat{a}_1 & 0 & 0 & 0 & 0 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 & \hat{b}_5 \\ 0 & \mu & -(\alpha + d) & 0 & 0 \\ \nu_2 & 0 & 0 & -(\theta + d) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{matrix}$$

$$\hat{a}_1 = (\mu + d)\nu\beta_1 - \frac{\beta_3(\beta + d)(\beta_2 - \mu - d)}{\alpha + d + \mu} - \beta_2(d + \nu_1 + \nu_2), \hat{b}_1 = \frac{\beta_3 - \beta_2}{\beta_2} \frac{(\alpha + d)(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

Where

$$\hat{b}_2 = -\frac{\beta_2(\alpha + d) + (d^2 + \alpha\mu + \mu d + \alpha d)}{\alpha + d + \mu},$$

$$\hat{b}_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}, \quad \hat{b}_4 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_5 = \frac{(\alpha + d)(\beta_2 - d - \mu)^2}{(\alpha + d + \mu)}$$

One of the eigen values is a_1 which is >0 i.e. E_2 is unstable

APPENDIX-C

Local stability of Equilibrium $E_3(Y_T^*, Y_H^*, Ya^*, Z^*, N^*)$ *By Lyapunov method,*

The model $(A_7 - A_{11})$ can be linearised by making of following assumptions

$$Y_T = Y_T^* + y_T, \quad Y_H = Y_H^* + y_H, \quad Y_a = Ya^* + y_a, \quad Z = Z^* + z, \quad N = N^* + n$$

we get

$$\frac{dY_T}{dt} = m_{11} y_T + m_{12} y_H + m_{13} y_a + m_{14} Z + m_{15} n$$

$$\frac{dY_H}{dt} = m_{21} y_T + m_{22} y_H + m_{23} y_a + m_{24} Z + m_{25} n$$

$$\frac{dy_a}{dt} = \mu y_H - (d + \alpha) y_a$$

$$\frac{dz}{dt} = \nu_2 y_T - (d + \theta) z$$

$$\frac{dn}{dt} = -\alpha Y_a - dn$$

Where we use following Lyapunov function ;

Where c_1, c_2, c_3, c_4, c_5 are to be determined.
So

$$\dot{V} = c_1 y_T \frac{dy_T}{dt} + c_2 y_H \frac{dy_H}{dt} + c_3 y_a \frac{dy_a}{dt} + c_4 z \frac{dz}{dt} + c_5 n \frac{dn}{dt}$$

$$= c_1 y_T (m_{11} y_T + m_{12} y_H + m_{13} y_a + m_{14} z + m_{15} n) + c_2 y_H (m_{21} y_T + m_{22} y_H + m_{23} y_a + m_{24} z + m_{25} n)$$

$$+ c_3 y_a (\mu y_H - (d+\alpha) y_a) + c_4 z (\nu_2 y_T - (\theta + d) z) + c_5 n (-\alpha y_a - dn)$$

$$= \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T y_H (c_1 m_{12} + c_2 m_{21}) + \frac{1}{4} c_2 m_{22} y_H^2\right] + \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T y_a (c_1 m_{13}) - \frac{1}{4} c_3 (d+\alpha) y_a^2\right]$$

$$+ \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T z (c_1 m_{14} + c_4 \nu_2) - \frac{1}{4} c_4 (d+\theta) z^2\right] + \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T n (c_1 m_{15}) - \frac{1}{4} c_5 d n^2\right]$$

$$+ \left[\frac{1}{4} c_2 m_{22} y_H^2 + y_H y_a (c_2 m_{23} + \mu c_3) - \frac{1}{4} c_3 (d+\alpha) y_a^2\right] + \left[\frac{1}{4} c_2 m_{22} y_H^2 + y_H z_2 (c_2 m_{24}) - \frac{1}{4} c_4 (d+\theta) z^2\right]$$

$$+ \left[\frac{1}{4} c_2 m_{22} y_H^2 + y_H n (c_2 m_{25}) - \frac{1}{4} c_5 d n^2\right] + \left[-\frac{1}{4} c_3 (d+\alpha) y_a^2 - 0 - \frac{1}{4} c_4 (d+\theta) z^2\right]$$

$$+ \left[-\frac{1}{4} c_3 (d+\alpha) y_a^2 + y_a n (-c_5 \alpha) - \frac{1}{4} c_5 d n^2\right] + \left[-\frac{1}{4} c_4 (d+\theta) z^2 + 0 - \frac{1}{4} c_5 d n^2\right]$$

for the equilibrium $E_3(Y_T^*, Y_H^*, Ya^*, Z^*, N^*)$

$$m_{11} = \frac{-\beta_1 Y_T^*}{N^*}, m_{12} = (-\beta_1 - \beta_3)\frac{Y_T^*}{N^*}, m_{13} = \frac{-\beta_1 Y_T^*}{N^*} = m_{14}, m_{15} = (\beta_1 - d - \nu_1 - \nu_2)\frac{Y_T^*}{N*}$$

$$m_{21} = (-\beta_2 + \beta_3)\frac{Y_H^*}{N*}, \; m_{22} = m_{23} = m_{24} = \frac{\beta_2 Y_H^*}{N^*}, \; m_{25} = ((\beta_2 - d - \mu))\frac{Y_H^*}{N*}$$

since all the coefficients of squre terms are negative so $\overset{0}{V}$ *WILL be negative definite provided the following conditions are satisfied.*

(i) $$[c_1(\beta_1 + \beta_3)\frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3)\frac{Y_H^*}{N^*}]^2 \langle \; \frac{c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{4N^{*2}}$$

(ii) $$c_1 \beta_1 Y_T^* \; \langle \; \frac{c_3 N^*(d+\alpha)}{4}$$

(iii) $$\left(\frac{c_1 \beta_1 Y_T^*}{N^*} - c_4 \nu_2\right)^2 \langle \; \frac{c_1 c_4 (\theta + d) \beta_1 Y_T^*}{N^*}$$

(iv) $$c_1 Y_T^* (\beta_1 - d - \nu_1 - \nu_2)^2 \langle \; \frac{c_5 \beta_1 N^* d}{4}$$

(v) $$[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3]^2 \; \langle \; \frac{c_2 c_3 (d+\alpha) \beta_2 Y_H^*}{4 N^*}$$

(vi) $$c_2 \beta_2 Y_H^* \; \langle \; \frac{c_4 (d+\theta) N^*}{4}$$

(vii) $c_2 Y_H^* (\beta_2 - d - \mu)^2 \langle \frac{c_5 d\beta_2 N^*}{4}$

(viii) $c_3 c_4 \rangle 0$

(ix) $c_5 \alpha^2 \langle \frac{c_3(d+\alpha)}{4}$

(x) $c_4 c_5 \rangle 0$

APPENDIX-D
(NON LINEAR STABILITY)
PROOF OF THE LEEMA

From model $(A_7 .... A_9)$ we can obtain as follows;

$$\frac{dN}{dt} = A - dN - \alpha Y_a \le A - dN \Rightarrow N \le \frac{A}{d}$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + \nu_1 + \nu_2) Y_T$$

$$\le (\beta_1 - d - \nu_1 - \nu_2) Y_T - \frac{\beta_1 Y_T^2}{N}$$

$$\Rightarrow Y_T \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d\beta_1}$$

similarly $Y_H \le \frac{(\beta_2 - d - \mu)A}{\beta_2 d}$ and $\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a$

$$\Rightarrow Y_a \le \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d(d+\alpha)}$$

$$\frac{dZ}{dt} = \nu_2 Y_T - dZ - \theta Z \Rightarrow Z \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)\nu_2}{d\beta_1 (d+\theta)}$$

so the region of attraction is

$0 \le Y_T \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d\beta_1}$, $0 \le Y_H \le \frac{(\beta_2 - d - \mu)A}{\beta_2 d}$,

$0 \le Y_a \le \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d(d+\alpha)}$, $0 \le Z \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)\nu_2}{d\beta_1 (d+\theta)}$

$0 \le N \le \frac{A}{d}$

PROOF OF THE THEOREM

In this case we use the following Lyapunov function;

$$W=\left(Y_T-Y_T^*-Y_T^*\log\frac{Y_T}{Y_T^*}\right)+m_1\left(Y_H-Y_H^*-Y_H^*\log\frac{Y_H}{Y_H^*}\right)+\frac{m_2}{2}\left(Y_a-Y_a^*\right)^2+\frac{m_3}{2}\left(N-N^*\right)^2+\frac{m_4}{2}\left(Z-Z^*\right)^2$$

on differentiating and substituting the values from model we get;

$$\left[\frac{-\beta_1}{4N^*}\left(Y_T-Y_T^*\right)^2+\left(Y_T-Y_T^*\right)\left(Y_H-Y_H^*\right)\left\{\frac{-\beta_1-\beta_3-\beta_2m_1+\beta_3m_1}{N^*}\right\}+\frac{1}{4}\left(Y_H-Y_H^*\right)^2\left(\frac{-\beta_2m_1}{N^*}\right)\right]$$

$$+\left[\frac{-\beta_1}{4N^*}\left(Y_T-Y_T^*\right)^2+\left(Y_T-Y_T^*\right)\left(Y_a-Y_a^*\right)\left(\frac{-\beta_2m_1}{N^*}+\mu m_2\right)-\frac{1}{4}m_2(d+\alpha)\left(Y_a-Y_a^*\right)^2\right]$$

$$+\left[\frac{-\beta_1}{4N^*}\left(Y_T-Y_T^*\right)^2+\left(Y_T-Y_T^*\right)\left(N-N^*\right)\left\{\frac{\beta_1Y_T+\beta_1Z+\beta_1Y_a+\left(\beta_1+\beta_3\right)Y_H}{NN^*}\right\}-\frac{1}{4}dm_3\left(N-N^*\right)^2\right]$$

$$+\left[\frac{-\beta_1}{4N^*}\left(Y_T-Y_T^*\right)^2+(Y_T-Y_T^*)(Z-Z^*)\left(\frac{-\beta_1}{N^*}+\nu_2m_4\right)-\frac{1}{4}m_4(d+\theta)(Z-Z^*)^2\right]$$

$$\left[\frac{1}{4}\left(Y_H-Y_H^*\right)^2\left(\frac{-\beta_2m_1}{N^*}\right)+\left(Y_H-Y_H^*\right)\left(Y_a-Y_a^*\right)\left\{\frac{-\beta_2m_1}{N^*}+\mu m_1\right\}-\frac{1}{4}m_2(d+\alpha)\left(Y_a-Y_a^*\right)^2\right]$$

$$+\left[\frac{1}{4}\left(Y_H-Y_H^*\right)^2\left(\frac{-\beta_2m_1}{N^*}\right)+\left(Y_H-Y_H^*\right)\left(N-N^*\right)\left\{\frac{\beta_2m_1Y_T+\left(Y_H+Y_a+Z\right)\beta_2m_1-\beta_3m_1Y_T}{NN^*}\right\}-\frac{1}{4}dm_3\left(N-N^*\right)^2\right]$$

$$+\left[\frac{1}{4}\left(Y_H-Y_H^*\right)^2\left(\frac{-\beta_2m_1}{N^*}\right)+\left(Y_H-Y_H^*\right)(Z-Z^*)\left(\frac{-\beta_2m_1}{N^*}\right)-\frac{1}{4}m_4(d+\theta)(Z-Z^*)^2\right]$$

$$+\left[-\frac{1}{4}m_2(d+\alpha)\left(Y_a-Y_a^*\right)^2+\left(Y_a-Y_a^*\right)\left(N-N^*\right)\left(-m_3\alpha\right)-\frac{1}{4}dm_3\left(N-N^*\right)^2\right]$$

$$+\left[-\frac{1}{4}m_2(d+\alpha)\left(Y_a-Y_a^*\right)^2+0-\frac{1}{4}m_4(d+\theta)(Z-Z^*)^2\right]+\left[-\frac{1}{4}dm_3\left(N-N^*\right)^2+0--\frac{1}{4}dm_3\left(N-N^*\right)^2\right]$$

CO
NSIDER $\frac{Y_T}{N},\frac{Y_H}{N},\frac{Y_a}{N},\frac{Z}{N}=1$

$\overset{0}{W}$ is negative definite if the following conditions are satisfied ;

$(i)\left(\beta_1+\beta_3+\beta_2m_1-\beta_3m_1\right)^2 \langle \frac{\beta_1\beta_2m_1}{4}$

(ii) $\left(\frac{\beta_2m_1}{N^*}-\mu m_2\right)^2 \langle \frac{\beta_1m_2(d+\alpha)}{4N^*}$

(iii) $(4\beta_1+\beta_3)^2 \langle \frac{dm_3\beta_1N^*}{4}$

(iv) $\left(\frac{-\beta_1}{N^*}+\nu_2m_4\right)^2 \langle \frac{m_4\beta_1(d+\theta)}{4N^*}$

(v) $\left(\frac{-\beta_2m_1}{N^*}+\mu m_2\right)^2 \langle \frac{m_1m_2(d+\alpha)\beta_2}{4N^*}$

(vi) $(4\beta_2 - \beta_3)^2 \frac{m_1}{N^*} \langle \frac{dm_3\beta_2}{4}$

(vii) $\left(\frac{\beta_2^2 m_1}{N^*}\right) \langle \frac{m_4(d+\theta)\beta_2}{4}$

(viii) $m_3\alpha^2 \langle \frac{m_2 d(d+\alpha)}{4}$

(ix) $m_2m_4, m_3m_4 \rangle 0$

FOR SIS MODEL;

APPENDIX-A1

The given model is

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + \nu_1 Y_T \quad ....(A_1)$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - \nu_1 Y_T \quad ....(A_2)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \quad ....(A_3)$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \quad ....(A_4)$$

Where $N = X + Y_T + Y_H + Y_a$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad ....(A_5)$$

the reduced system is given below,

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + \nu_1)Y_T ....(A_6)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_a - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu)Y_H ....(A_7)$$

$$\frac{dY_a}{dt} = \mu Y_H - (d + \alpha)Y_a ....(A_8)$$

$$\frac{dN}{dt} = A - dN - \alpha Y_a ....(A_9)....$$

EXISTENCE OF $E_0(0,0,0,\frac{A}{d})$

It is obvious from the model $(A_6..........A_9)$,we can find if $Y_T = Y_H = Y_a = 0$

So $N = \frac{A}{d}$

This is diease free equilibrium.

EXISTENCE OF $E_1(\overline{Y}_T, 0, 0, \overline{N})$

Since $Y_H = 0 \Rightarrow Y_a = 0$ and $Y_T \neq 0$

From (A_9) $we \; get \; \overline{N} = \frac{A}{d},$

from (A_6) $we \; get$

$$\beta_1 Y_T (\overline{N} - \overline{Y}_T) - (d + \nu_1) \overline{Y}_T \overline{N} = 0$$

Since $\overline{Y}_T \neq 0$

So on solving , we get

$$\overline{Y}_T = \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d\beta_1 \nu}, \overline{N} = \frac{A}{d}$$

Which exists if $\beta_1 \rangle d + \nu_1$

EXISTENCE OF $E_2(0, \hat{Y}_H, \hat{Y}_a, \hat{N})$

Since $\hat{Y}_T = 0 \, And \; \hat{Y}_H, \hat{Y}_a \neq 0$

So ,on solving A_7 , A_8 , A_9 We get,

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$, Which exists if $\beta_2 \rangle d + \mu$

EXISTENCE OF $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$

(A_8), $we \; get \; Y_a^* = \frac{\mu Y^*{}_H}{d + \alpha}$,

From $From \; (A_9) \quad N^* = \frac{A}{d} - \frac{\alpha\mu Y_H^*}{d(d + \alpha)}$

So $A_6 \; and \; A_7 \; Becomes$

$$\beta_1 Y^*{}_T + Y_H^* \left[\frac{\alpha\mu(\beta_1 - d - \nu_1)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1 \mu}{d + \alpha} + \beta_3 \right] = \frac{A(\beta_1 - d - \nu_1)}{d} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A_{10})$$

$$Y_T^*(\beta_2 - \beta_3) + Y_H^* \left[\frac{\alpha\mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha} \right] = \frac{A(\beta_2 - d - \mu)}{d} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A_{11})$$

$where$

$$a_1 = \beta_1, \; a_2 = \frac{\alpha\mu(\beta_1 - d - \nu_1)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1 \mu}{d + \alpha} + \beta_3, \; a_3 = \beta_2 - \beta_3$$

$$a_4 = \frac{\alpha\mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha}, \; b_1 = \frac{A(\beta_1 - d - \nu_1)}{d}, \; b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

so $A_{12} \; And \; A_{13} \; Becomes$

$a_1Y_T^* + a_2Y_H^* = b_1 \ , a_3Y_T^* + a_4Y_H^* = b_2 ....(A_{12})$

now here arise the three cases –

(1);if $a_3 = 0 \Rightarrow \beta_2 = \beta_3$ *so the equilibrium* $E_{30}(Y_{TO}^*, Y_{HO}^*, Y_{a0}^*, N_0^*)$
so

$$Y_{TO}^* = \frac{b_1a_4 - a_2b_2}{a_1a_4} \ , \ Y_{H0}^* = \frac{b_2}{a_4}, \ Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}$$

$$Z_0^* = \frac{\nu_2(b_1a_4 - a_2b_2)}{a_1a_4}, \ N_0^* = \frac{A}{d} - \frac{\alpha}{d}\frac{\mu b_2}{a_4(\alpha + d)}$$

$$\beta_2 \rangle d + \mu$$

it exists only when $\left[\{\beta_1(d+\mu) - \beta_2(d+\nu_1)\}(1+\frac{\mu}{\alpha+d}) - \beta_3(\beta_2 - d - \mu) \rangle 0\right]$

(2)if $a_3 < 0$ so The Equilibrium $E_{31}(Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, N_1^*)$, *Where*

$$Y_{T1}^* = \frac{a_4b_1 - a_2b_2}{a_1a_4 + a_2a_3} \ , Y_{H1}^* = \frac{a_1b_2 + a_3b_3}{a_1a_4 + a_2a_3} \ ,$$

$N_1^* = \frac{A}{d} - \frac{\alpha\mu Y_{H1}^*}{d(d+\alpha)}$, *it exists providedthe following conditions are satisfied.*

$$[\beta_1(d+\mu) - \beta_2(d+\nu_1)](1+\frac{\mu}{d+\alpha}) - \beta_3(\beta_2 - d - \mu)\rangle 0$$

$$\beta_1\beta_2\left[\frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d+\alpha}\right] - \beta_3(\beta_2 - d - \mu)\frac{\alpha\mu}{d} - \mu(d+\alpha)\beta_2 \ \rangle 0$$

(3) if $a_3 > 0$ *soThe Equilibrium* © $E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, N_2^*)$,(*THE GENERAL CASE*)
Where

$$Y_{T2}^* = \frac{a_4b_1 - a_2b_2}{a_1a_4 - a_2a_3}, \ Y_{H2}^* = \frac{a_1b_2 - a_3b_1}{a_1a_4 - a_2a_3}, \ Y_{a2}^* = \frac{\mu Y_{H2}^*}{d+\alpha}$$

$N_2^* = \frac{A}{d} - \frac{\alpha\mu Y_{H2}^*}{d(d+\alpha)}$, *it exists provided that*

$a_4b_1 \rangle \ a_2b_2$, *and* $a_1b_2 \rangle a_3b_1$,*i.e.*

$$[\beta_1(d+\mu) - \beta_2(d+\nu_1)](1+\frac{\mu}{d+\alpha}) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - \nu_1)}{\beta_1} - (d+\mu) \ \rangle 0 \ ,$$

APPENDIX-B1

Using the model $(A_6 ....A_9)$, The general jacobian matrix can be written as follows;

$$M = \begin{array}{cccc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & \mu & -(\alpha + d) & 0 \\ 0 & 0 & -\alpha & -d \end{array}$$

where

$$m_{11} = \frac{\beta_1 (N - Y_T - Y_H - Y_a)}{N} - \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_H}{N} - d - \nu_1 \; m_{12} = \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_T}{N}, m_{13} = \frac{-\beta_1 Y_T}{N}$$

$$m_{14} = \frac{\beta_1 Y_T}{N} - \frac{\beta_1 Y_T \left(N - Y_T - Y_H - Y_a\right)}{N^2} + \frac{\beta_3 Y_T Y_H}{N^2}$$

$$m_{21} = \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_H}{N}, m_{22} = \frac{\beta_2 \left(N - Y_T - Y_H - Y_a - Z\right)}{N} - \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_T}{N} - (d + \mu)$$

$$m_{23} = \frac{-\beta_2 Y_H}{N}$$

$$m_{24} = \frac{\beta_2 Y_H}{N} - \frac{\beta_2 Y_H (N - Y_T - Y_H - Y_a)}{N^2} - \frac{\beta_3 Y_T Y_H}{N^2}$$

PROOF OF THEOREM-1; stability of $E_0(0,0,0,\frac{A}{d})$

The jacobian matrix for E_0 is given by

$$M(E_0) = \begin{array}{cccc} \beta_1 - d - \nu_1 - \nu_2 & 0 & 0 & 0 \\ 0 & \beta_2 - d - \mu & 0 & 0 \\ 0 & \mu & -(d + \alpha) & 0 \\ 0 & 0 & -\alpha & -d \end{array}$$

the eigen values are $\beta_1 - d - \nu_1$, $\beta_2 - d - \mu$, $-(d+\alpha)$, $-(d+\theta)$,-d

since $\beta_1 \rangle \, d + \nu_1$, $\beta_2 \rangle \, d + \mu$ so the equilibrium E_0 is unstable.

Stability for $E_1(\bar{Y}_T, 0, 0, \bar{N})$

The jacobian matrix for E_1 is given as

$$M(E_1) = \begin{array}{cccc} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & 0 & 0 \\ 0 & \mu & -(\alpha + d) & 0 \\ 0 & 0 & -\alpha & -d \end{array}$$

where $b_{11} = -(\beta_1 - d - \nu_1), b_{12} = -(\beta_1 + \beta_3)(\beta_1 - d - \nu_1), b_{13} = -(\beta_1 - d - \nu_1)$

$$b_{14} = \frac{(\beta_1 - d - \nu_1)}{\beta_1} d^2$$

$$b_{22} = \frac{\beta_2}{\beta_1} d + \frac{\beta_3}{\beta_1} (\beta_1 - d - \nu_1) - (d + \mu)$$

Since eigen value are , $b_{11}, b_{22}, -(\alpha + d), -(\theta + d), -d$

Since $b_{22} > 0$ *so* E_1 *is unstable*

stability of $E_2(0, \hat{Y}_H, \hat{Y}_a, \hat{N})$,

The jacobian for E_2 *is*

$$M(E_2) = \begin{matrix} \hat{a}_1 & 0 & 0 & 0 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 \\ 0 & \mu & -(\alpha + d) & 0 \\ 0 & 0 & -\alpha & -d \end{matrix}$$

Where $\hat{a}_1 = (\mu + d)\beta_1 - \frac{\beta_3(\beta + d)(\beta_2 - \mu - d)}{\alpha + d + \mu} - \beta_2(d + \nu_1), \hat{b}_1 = \frac{\beta_3 - \beta_2}{\beta_2} \frac{(\alpha + d)(\beta_2 - d - \mu)}{\alpha + d + \mu}$

$$\hat{b}_2 = -\frac{\beta_2(\alpha + d) + (d^2 + \alpha\mu + \mu d + \alpha d)}{\alpha + d + \mu},$$

$$b_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}, \quad \hat{b}_5 = \frac{(\alpha + d)(\beta_2 - d - \mu)^2}{(\alpha + d + \mu)}$$

One of the eigen values is a_1 which is >0 i.e. E_2 is unstable

APPENDIX-C1

Local stability of Equilibrium $E_3(Y_T^*, Y_H^*, Ya^*, N^*)$ *By Lyapunovmethod,*

The model $(A_6 - A_9)$ can be linearised by making of following assumptions

$$Y_T = Y_T^* + y_T, \quad Y_H = Y_H^* + y_H, \quad Y_a = Ya^* + y_a, \; N = N^* + n$$

we get

$$\frac{dY_T}{dt} = m_{11} y_T + m_{12} y_H + m_{13} y_a + m_{14} n$$

$$\frac{dY_H}{dt} = m_{21} y_T + m_{22} y_H + m_{23} y_a + m_{24} n$$

$$\frac{dy_a}{dt} = \mu y_H - (d+\alpha)y_a$$

$$\frac{dn}{dt} = -\alpha Y_a - dn$$

Where we use following Lyapunov function ;

$$V = \frac{c_1 y_T^2}{2} + \frac{c_2 y_H^2}{2} + \frac{c_3 y_a^2}{2} + \frac{c_4 n^2}{2}$$

Where c_1, c_2, c_3, c_4 are to be determined.

So

$$\dot{V} = c_1 y_T \frac{dy_T}{dt} + c_2 y_H \frac{dy_H}{dt} + c_3 y_a \frac{dy_a}{dt} + c_4 n \frac{dn}{dt}$$

$$= c_1 y_T (m_{11} y_T + m_{12} y_H + m_{13} y_a + m_{14} n) + c_2 y_H (m_{21} y_T + m_{22} y_H + m_{23} y_a + m_{24} n)$$
$$+ c_3 y_a (\mu y_H - (d+\alpha) y_a) + c_4 n(-\alpha y_a - dn)$$

$$= \left[\frac{1}{3} c_1 m_{11} y_T^2 + y_T y_H (c_1 m_{12}) + \frac{1}{3} c_2 m_{22} y_H^2\right] + \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T y_a (c_1 m_{13}) - \frac{1}{4} c_3 (d+\alpha) y_a^2\right]$$
$$+ \left[\frac{1}{3} c_1 m_{11} y_T^2 + y_T n (c_1 m_{14}) - \frac{1}{3} c_4 dn^2\right]$$
$$+ \left[\frac{1}{3} c_2 m_{22} y_H^2 + y_H y_a (c_2 m_{23} + \mu c_3) - \frac{1}{4} c_3 (d+\alpha) y_a^2\right]$$
$$+ \left[\frac{1}{3} c_2 m_{22} y_H^2 + y_H n (c_2 m_{24}) - \frac{1}{3} c_4 dn^2\right]$$
$$+ \left[-\frac{1}{3} c_3 (d+\alpha) y_a^2 + y_a n(-c_4 \alpha) - \frac{1}{3} c_4 dn^2\right]$$

for the equilibrium $E_3(Y_T^*, Y_H^*, Ya^*, N^*)$

$$m_{11} = \frac{-\beta_1 Y_T^*}{N^*}, m_{12} = (-\beta_1 - \beta_3)\frac{Y_T^*}{N^*}, m_{13} = \frac{-\beta_1 Y_T^*}{N^*}, m_{14} = (\beta_1 - d - \nu_1)\frac{Y_T^*}{N*}$$

$$m_{21} = (-\beta_2 + \beta_3)\frac{Y_H^*}{N*},\ m_{22} = m_{23} = \frac{\beta_2 Y_H^*}{N^*},\ m_{24} = ((\beta_2 - d - \mu))\frac{Y_H^*}{N*}$$

since all the coefficients of squre terms are negative so $\overset{0}{V}$ *WILL be negative definite provided the following conditions are satisfied.*

(i) $$[c_1(\beta_1 + \beta_3)\frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3)\frac{Y_H^*}{N^*}]^2 \langle \frac{4 c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{9 N^{*2}}$$

(ii) $$c_1 \beta_1 Y_T^* \langle \frac{4 c_3 N^* (d+\alpha)}{9}$$

(iii) $c_1 Y_T^* (\beta_1 - d - \nu_1)^2 \langle \frac{4 c_4 \beta_1 N^* d}{9}$

(iv) $[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3]^2 \langle \frac{4 c_2 c_3 (d+\alpha) \beta_2 Y_H^*}{9 N^*}$

(v) $c_2 Y_H^* (\beta_2 - d - \mu)^2 \langle \frac{4 c_4 d \beta_2 N^*}{9}$

(vi) $c_4 \alpha^2 \langle \frac{4 c_3 d (d+\alpha)}{9}$

APPENDIX-D1
(NON LINEAR STABILITY)
PROOF OF THE LEEMA

From model $(A_6 .... A_9)$ we can obtain as follows;

$$\frac{dN}{dt} = A - dN - \alpha Y_a \le A - dN \Rightarrow N \le \frac{A}{d}$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d+\nu_1) Y_T$$

$$\le (\beta_1 - d - \nu_1) Y_T - \frac{\beta_1 Y_T^2}{N}$$

$$\Rightarrow Y_T \le \frac{A(\beta_1 - d - \nu_1)}{d \beta_1}$$

similarly $Y_H \le \frac{(\beta_2 - d - \mu) A}{\beta_2 d}$ and $\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \Rightarrow Y_a \le \frac{(\beta_2 - d - \mu) A \mu}{\beta_2 d (d+\alpha)}$

so the region of attraction is

$0 \le Y_T \le \frac{A(\beta_1 - d - \nu_1 - \nu_2)}{d \beta_1}$, $0 \le Y_H \le \frac{(\beta_2 - d - \mu) A}{\beta_2 d}$, $0 \le Y_a \le \frac{(\beta_2 - d - \mu) A \mu}{\beta_2 d (d+\alpha)}$,

$0 \le N \le \frac{A}{d}$

PROOF OF THE THEOREM

In this case we use the following Lyapunov function;

$$W = \left(Y_T - Y_T^* - Y_T^* \log \frac{Y_T}{Y_T^*} \right) + m_1 \left(Y_H - Y_H^* - Y_H^* \log \frac{Y_H}{Y_H^*} \right) + \frac{m_2}{2} \left(Y_a - Y_a^* \right)^2 + \frac{m_3}{2} \left(N - N^* \right)^2$$

on differentiating and substituting the values from model we get;

$$\left[\frac{-\beta_1}{3N^*} \left(Y_T - Y_T^* \right)^2 + \left(Y_T - Y_T^* \right) \left(Y_H - Y_H^* \right) \left\{ \frac{-\beta_1 - \beta_3 - \beta_2 m_1 + \beta_3 m_1}{N^*} \right\} + \frac{1}{3} \left(Y_H - Y_H^* \right)^2 \left(\frac{-\beta_2 m_1}{N^*} \right) \right]$$

$$+ \left[\frac{-\beta_1}{3N^*} \left(Y_T - Y_T^* \right)^2 + \left(Y_T - Y_T^* \right) \left(Y_a - Y_a^* \right) \left(\frac{-\beta_2 m_1}{N^*} \right) - \frac{1}{4} m_2 (d + \alpha) \left(Y_a - Y_a^* \right)^2 \right]$$

$$+ \left[\frac{-\beta_1}{3N^*} \left(Y_T - Y_T^* \right)^2 + \left(Y_T - Y_T^* \right) \left(N - N^* \right) \left\{ \frac{\beta_1 Y_T + \beta_1 Y_a + (\beta_1 + \beta_3) Y_H}{NN^*} \right\} - \frac{1}{3} d m_3 \left(N - N^* \right)^2 \right] +$$

$$\left[\frac{1}{3} \left(Y_H - Y_H^* \right)^2 \left(\frac{-\beta_2 m_1}{N^*} \right) + \left(Y_H - Y_H^* \right) \left(Y_a - Y_a^* \right) \left\{ \frac{-\beta_2 m_1}{N^*} + \mu m_2 \right\} - \frac{1}{3} m_2 (d + \alpha) \left(Y_a - Y_a^* \right)^2 \right]$$

$$+ \left[\frac{1}{3} \left(Y_H - Y_H^* \right)^2 \left(\frac{-\beta_2 m_1}{N^*} \right) + \left(Y_H - Y_H^* \right) \left(N - N^* \right) \left\{ \frac{\beta_2 m_1 Y_T + (Y_H) \beta_2 m_1 - \beta_3 m_1 Y_T}{NN^*} \right\} - \frac{1}{3} d m_3 \left(N - N^* \right)^2 \right]$$

$$+ \left[-\frac{1}{3} m_2 (d + \alpha) \left(Y_a - Y_a^* \right)^2 + \left(Y_a - Y_a^* \right) \left(N - N^* \right) \left(-m_3 \alpha \right) - \frac{1}{3} d m_3 \left(N - N^* \right)^2 \right]$$

CONSIDE

$$\frac{Y_T}{N}, \frac{Y_H}{N}, \frac{Y_a}{N} = 1$$

$\overset{0}{W}$ is negative definite if the following conditions are satisfied ;

(i) $(\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 \langle \frac{4\beta_1 \beta_2 m_1}{9}$

(ii) $\beta_1 \langle \frac{4 m_2 (d + \alpha) N^*}{9}$

(iii) $(3\beta_1 + \beta_3)^2 \langle \frac{4 d m_3 \beta_1 N^*}{9}$

(iv) $\left(\frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 \langle \frac{4 m_1 m_2 (d + \alpha) \beta_2}{9 N^*}$

(v) $(\beta_2 m_1 + \beta_2 m_2 + \beta_3 m_1)^2 \langle \frac{4 \beta_2 m_1 m_3 d N^*}{9}$

(iv) $m_3 \alpha^2 \langle \frac{4 m_2 d (d + \alpha)}{9}$

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