A STUDY ON OSCILLATION OF NONLINEAR NEUTRAL IMPULSIVE HYPERBOLIC DIFFERENTIAL EQUATION USING ROBIN BOUNDARY CONDITION

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Abstract: In this paper, we consider the forced oscillation of nonlinear neutral type impulsive hyperbolic differential equation with several delays under the Robin boundary condition are investigated and several new sufficient conditions for oscillation are presented.

1. INTEODUCTION

The problem of oscillation and nonoscillation of solution of impulsive partial differential equations has attracted a great deal of attention over the last few years. The first paper on impulsive partial differential equation [10] was published in 1991. For an excellent exposition of this paper and its applications, see [2,3,7,14,23,29].

The oscillation theory of impulsive partial differential equations can be applied to many fields, such as to biology, engineering, medicine, physics and chemistry. Recently, the theory of impulsive partial differential equations has been investigated by many authors [4-6,8,11,24,30]. However , very few attention has been given to impulsive partial differential equations with delay [1,9,12,13,15-17,19-22,25-28,31], especially impulsive partial differential equations of neutral type.

In this paper, we study the forced oscillation of nonlinear neutral type impulsive hyperbolic differential equation (1.1) under the boundary condition (1.2),(1.3) for the oscillation solutions of the equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}[u(x,t)+g(t)u(x,t-\tau)]\right)=a(t)\Delta u(x,t)-p(x,t)f(u(x,t-\sigma))\\
&-\sum_{j=1}^{n}q_j(x,t)f_j(u(x,t-\mu_j))+F(x,t),t\neq t_k,(x,t)\in\Omega\times\mathbb{R}^{+}=G \qquad (1.1)\\
&u(x,t_k^{+})=\alpha_k(x,t_k,u(x,t_k)),\\
&u(x,t_k^{+})=\beta_k(x,t_k,u(x,t_k)),t=t_k,k=1,2,3,
\end{aligned}
$$

With the boundary conditions

$$\frac{\partial u}{\partial \gamma}+h(x,t)u=0,(x,t)\in\partial\Omega\times\mathbb{R}^{+} \tag{1.2}$$

$$u=0,\in\partial\Omega\times\mathbb{R}^{+} \tag{1.3}$$

And the initial condition

$$u(x,t)=\phi(x,t),(x,t)\in[-\delta,0]\times\Omega. \tag{1.4}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and Δ is the Laplacian in the Euclidean N-space $\mathbb{R}^N$, γ is a unit exterior normal vector of $\partial\Omega$, $\delta=\max\{\tau,\sigma,\mu_j\},\phi(x,t)\in C^2([-\delta,0]\times\Omega,\mathbb{R})$. In the sequel we assume that the following conditions are fulfilled:

(H1) $0<t_1<t_2<\ldots<t_k<\ldots,\lim_{k\to\infty}t_k=\infty, g\in PC(\mathbb{R}^{+},\mathbb{R}^{+}), 0<g(t)<1$ and g is increasing function,are constants which are greater than zero, $j\in I_n$.

(H2) $r(t),a(t)\in PC(\mathbb{R}^{+},\mathbb{R}^{+}), p(x,t),q_j(x,t)$ are class of functions which are positive piecewise continuous in t with discontinuous of first kind only at $t=t_k,k=1,2,3,\ldots$ and left continuous at $t=t_k,k=1,2,3,\ldots$

(H3) $h(x,t)\in PC(\partial\Omega\times\mathbb{R}^{+},\mathbb{R}^{+}), f(u), f_j(u)\in C(\mathbb{R}^{+},\mathbb{R}^{+}); \frac{f(u)}{u}\geq C$ is a positive constant, $\frac{f_j(u)}{u}\geq C_j$ is a positive constant, for $u\neq 0; f(-u)=-f(u)$ and $f_j(-u)=-f_j(u), j\in I_n; \int_\Omega F(x,t)dx\leq 0.$

(H4) $u(x,t)$ and their derivatives $u_t(x,t)$ are piecewise continuous in t with discontinuous of first kind only at $t=t_k,k=1,2,3,\ldots$ and left continuous at $t=t_k, u(x,t_k)=u(x,t_k^{-}), u_t(x,t_k)=u_t(x,t_k^{-}), k=1,2,3,\ldots$

(H5) $\alpha_k(x,t_k,u(x,t_k)),\beta_k(x,t_k,u_t(x,t_k))\in PC(\mathbb{R}^{+}\times\overline{\Omega}\times\mathbb{R},\mathbb{R}), k=1,2,3,\ldots,$ and there exist positive constants a_k^*,b_k,b_k^* and $b_k\leq a_k^*$ such that for $k=1,2,3,\ldots,$

$$a_k^*\leq\frac{\alpha_k(x,t_k,\xi)}{\xi}\leq a_k$$

$$b_k^*\leq\frac{\beta_k(x,t_k,\xi)}{\xi}\leq b_k$$

Let us construct the sequence $\{\bar{t}_k\}=\{t_k\}\cup\{t_{k\tau}\}\cup\{t_{k\sigma}\}\cup\{t_{k\mu_j}\}$, where

$t_{k\tau}=t_k+\tau, t_{k\sigma}=t_k+\sigma, t_{k\mu_j}=t_k+\mu_j, j\in I_n$ and $\bar{t}_k<\bar{t}_{k+1}, k=1,2,3,\ldots$

By a solution of problem (1.1),(1.2)((1.1),(1.3)) with initial condition (1.4), we mean that any function $u(x,t)$ for which the following conditions are valid:

(1) If $-\delta\leq t\leq 0$, then $u(x,t)=\phi(x,t)$.

(2) If $0\leq t\leq\bar{t}_1=t_1$, then $u(x,t)$ coincides with the solution of the problem (1.1)and (1.2)((1.3)) with initial condition.

(3) If $\bar{t}_k<t\leq\bar{t}_{k+1}, \bar{t}_k\in\{t_k\}\setminus\{t_{k\tau}\}\cup\{t_{k\sigma}\}\cup\{t_{k\mu_j}\}$, then $u(x,t)$ coincides with the solution of the problem (1.1)and (1.2)((1.3)).

(4) If $\bar{t}_k < t \le \bar{t}_{k+1}, \bar{t}_k \in \{t_{k\tau}\} \cup \{t_{k\sigma}\} \cup \{t_{k\mu_j}\}$, then $u(x,t)$ coincides with the solution of the problem (1.2)((1.3))and the following equations

$$\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}[u(x,t^{+}) + g(t)u(x,(t-\tau)^{+})]\right) = a(t)\Delta u(x,t^{+}) - p(x,t)f(u(x,(t-\sigma)^{+}))$$

$$-\sum_{j=1}^{n} q_j(x,t) f_j(u(x,(t-\sigma_j)^{+})) + F(x,t), t \ne t_k, (x,t) \in \Omega \times \mathbb{R}^{+} = G$$

$$u(x,\bar{t}_k^{+}) = u(x,\bar{t}_k), u_t(x,\bar{t}_k^{+}) = u_t(x,\bar{t}_k), \bar{t}_k \in \{t_k\} \setminus \{t_{k\tau}\} \cup \{t_{k\sigma}\} \cup \{t_{k\mu_j}\},$$

Or

$$u(x,t_k^{+}) = \alpha_k(x,\bar{t}_k, u(x,\bar{t}_k)),$$
$$u_t(x,t_k^{+}) = \beta_k(x,\bar{t}_k, u(x,\bar{t}_k)), \bar{t}_k \in \{t_{k\tau}\} \cup \{t_{k\sigma}\} \cup \{t_{k\mu_j}\} \cap \{t_k\}.$$

Here the number k_i is determined by the equality $\bar{t}_k = t_{k_i}$.

We introduce following the notations throughout this paper:

$$\Gamma_k = \{(x,t) : t \in (t_k, t_{k+1}), x \in \Omega\}; \Gamma = \bigcup_{k=0}^{\infty} \Gamma_k,$$
$$\overline{\Gamma}_k = \{(x,t) : t \in (t_k, t_{k+1}), x \in \overline{\Omega}\}; \overline{\Gamma} = \bigcup_{k=0}^{\infty} \overline{\Gamma}_k,$$
$$p(t) = \min_{x\in\overline{\Omega}} p(x,t), q_j(t) = \min_{x\in\overline{\Omega}} q_j(x,t),$$
$$v(t) = \int_{\Omega} u(x,t)dx,$$

and

$$z(t) = v(t) + g(t)v(t-\tau)$$

The solution $u \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ of problem (1.1),(1.2) ((1.1),(1.3)) is called nonoscillatory in the domain G if it either eventually positive or eventually negative. Otherwise, it is called oscillatory.

This paper is organized as follows: Section 2, deals with the oscillatory properties of solutions for problems (1.1) and (1.2). In Section 3, we discuss the oscillatory properties of solutions for problems (1.1) and (1.3). Section 4 , deals with an example to illustrate the main results.

2. OSCILLATION PROPERTIES OF THE PROBLEM (1.1) AND (1.2)

To prove the main result, we need the following lemmas,

Lemma 2.1. Let $u \in C^2(\Gamma) \cap C^1(\overline{\Gamma})$ be a positive solution of the problem (1.1),(1.2) in the domain G , then the function satisfies the impulsive differential inequality

$$r(t)z^{"}(t)+C(1-g(t))p(t)z(t-\sigma)(1-g(t))\sum_{j=1}^{n}C_jq_j(t)z(t-\mu_j)\le 0, t\ne t_k \quad (2.1)$$

$$a_k^*\le \frac{z(\bar{t_k})}{z(t_k)}\le a_k \quad (2.2)$$

$$b_k^*\le \frac{z^{'}(\bar{t_k})}{z^{'}(t_k)}\le b_k, t=t_k, k=1,2,3,... \quad (2.3)$$

Proof. Let $u(x,t)$ be a positive solution of the problem (1.1),(1.2) in G. Without loss of generality, we may assume that there exists a $T>0$, where $t_0>T$ such that

$$u(x,t)>0, u(x,t-\tau)>0, u(x,t-\sigma)>0, u(x,t-\mu_j)>0,\ j=1,2,3,...,n.$$

For $t\ge t_0, t\ne t_k, k=1,2,3,...,$ integrating (1.1) with respect to x, over Ω yields

$$\frac{d}{dt}\left(r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx+g(t)\int_\Omega u(x,t-\tau)dx\right)\right)=a(t)\int_\Omega \Delta u(x,t)dx-\int_\Omega p(x,t)f(u(x,(t-\sigma))dx$$
$$-\sum_{j=1}^{n}\int_\Omega q_j(x,t)f_j(u(x,(t-\mu_j)dx+\int_\Omega F(x,t)dx. \quad (2.4)$$

By Green's formula and the boundary condition (1.2) we have

$$\int_\Omega \Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u}{\partial\gamma}dS=-\int_{\partial\Omega}h(x,t)u(x,t)dS\le 0 \quad (2.5)$$

Where dS is the surface element on $\partial\Omega$.

Also from condition (H3) and Jenson's inequality we can easily obtain

$$\int_\Omega p(x,t)f(u(x,(t-\sigma))dx\ge p(t)f\left(\int_\Omega(u(x,(t-\sigma))\right)dx,$$
$$\ge p(t)f(v(t-\sigma)) \quad (2.6)$$
$$\ge Cp(t)v(t-\sigma)$$
$$\int_\Omega q_j(x,t)f_j(u(x,(t-\mu_j)dx\ge C_jq_j(t)v(x,t-\mu_j)$$

Combining (2.4)-(2.6), we get

$$r(t)(v(t)+g(t)v(t-\tau))^{"}+Cp(t)v(t-\sigma)+\sum_{j=1}^{n}C_jq_j(t)v(x,t-\mu_j)\le 0 \quad (2.7)$$

Let $z(t)=v(t)+g(t)v(t-\tau)$, then $z(t)>0, z(t)\geq v(t)$. By the inequality (2.7), $z^{"}(t)\leq 0, t\geq t_0$ and it is easy to obtain

$$z^{'}(t)\geq 0, t\geq t_0. \qquad (2.8)$$

In fact, if the inequality (2.8) doesnot bounded, there exists $t_1\geq t_0$ satisfying $z_1^{'}(t)\leq 0$. Since $z^{'}(t)$ is decreasing, then

$$z(t)-z(t_1)=\int_{t_1}^{t} z^{'}(s)ds\leq\int_{t_1}^{t} z^{'}(t_1)ds=z_1^{'}(t)(t-t_1)$$

and $\lim_{t\to\infty} z(t)=-\infty$, which contradicts $z(t)>0$, so (2.8)holds. From (2.7), we have

$$r(t)z^{"}(t)+Cp(t)v(t-\sigma)+\sum_{j=1}^{n}C_jq_j(t)v(t-\mu_j)\leq 0.$$

Because,

$$\begin{aligned}
&v(t)=z(t)-g(t)v(t-\tau)\\
&=z(t)-g(t)[z(t-\tau)v(t-2\tau)]\\
&=z(t)-g(t)z(t-\tau)+g(t)g(t-\tau)v(t-2\tau)\\
&\geq z(t)-g(t)z(t-\tau)\\
&=(1-g(t))z(t)
\end{aligned}$$

So,

$$\begin{aligned}
&v(t-\sigma)\geq(1-g(t-\sigma))z(t-\sigma)\\
&\geq 91-g(t))z(t-\sigma)\\
&v(t-\mu_j)\geq(1-g(t-\mu_j))z(t-\mu_j)\\
&\geq(1-g(t)z(t-\mu_j), j=1,2,3,...,n
\end{aligned}$$

Hence, we obtain

$$r(t)z^{"}(t)+(1-g(t)Cp(t)z(t-\sigma)+\sum_{j=1}^{n}C_jq_j(t)(1-g(t))z(t-\mu_j)\leq 0.t\neq t_k$$

For $t\geq t_0, t=t_k, k=1,2,...,$ from (1.1) and the condition (H5),we obtain

$$a_k^*\leq\frac{u(x,t_k^+)}{u(x,t_k)}\leq a_k$$

$$b_k^*\leq\frac{u_t(x,t_k^+)}{u_t(x,t_k)}\leq b_k.$$

According to the $v(t)=\int_{\Omega}u(x,t)dx,$, we obtain

$$a_k^* \le \frac{v(t_k^+)}{v(t_k)} \le a_k$$

$$b_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le b_k, k = 1,2,3,...$$

Because $z(t) = v(t) + g(t)v(t-\tau),$ we have

$$a_k^* \le \frac{z(t_k^+)}{z(t_k)} \le a_k$$

$$b_k^* \le \frac{z'(t_k^+)}{z'(t_k)} \le b_k, k = 1,2,3,...$$

Hence, $z(t)$ is a positive solution of impulsive differential (2.1)-(2.3), This completes the proof.

Lemma 2.2. [14] Assume that

(A1) the sequence $\{t_k\}$ satisfies $0 < t_0 < t_1 < ..., \lim_{k\to\infty} t_k = \infty$;

(A2) $m(t) \in PC'[\mathbb{R}^+, \mathbb{R}]$ is left continuous at t_k for $k = 1,2,3,...$

(A3) for $k = 1,2,3,...$ and $t \ge t_0$,

$$m'(t) \le p(t)m(t) + q(t), t \ne t_k$$
$$m(t_k^+) \le d_k m(t_k) + e_k$$

Where $p(t), q(t) \in C(\mathbb{R}^+, \mathbb{R}), d_k \ge 0$ and e_k are constants. PC denote the class of piecewise continuous function from $\mathbb{R}^+$ to $\mathbb{R}$,with discontinuous of the first kind only at $t = t_k k = 1,2,...$

Then

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds)\right) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(r)dr)\right) q(s)ds$$
$$+ \sum_{t_0<t_k<t} \prod_{t_0<t_k<t} d_j \exp\left(\int_{t_0}^{t} p(s)ds\right) e_k.$$

Lemma 2.3. Let $z(t)$ be an eventually positive (negative) solution of the differential inequality (2.1)-(2.3). Assume that there exists $T \ge t_0$ such that $z(t) > 0 (z(t) < 0)$ for $t \ge T$. if

$$\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{b_k^*}{a_k} ds = +\infty \tag{2.9}$$

Hold, then $z'(t) \ge 0 (z'(t) \le 0)$ for $t \in [T, t_1] \cup \left(\bigcup_{k=l}^{+\infty} (t_k, t_{k+1}] \right)$ where $l = \min\{k : t_k \ge T\}$.

Proof .The proof of the lemma can be found in [18].

We begin with the following theorem.

Theorem 2.1. If condition (2.9) and the following condition hold

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{a_k^*}{b_k}r(t_k)\psi(s)ds=+\infty, \tag{2.10}$$

Where,

$$\psi(t)=\frac{\exp(-\delta w(t_0))(1-g(t))\left[Cp(t)+\sum_{j=1}^{n}C_jq_j(t)\right]}{r(t)}$$

Then every solution of the problem (1.1),(1.2) oscillates in G .

Proof. Let $u(x,t)$ be a nonoscillatory solution of (1.1),(1.2). Without loss of generality, we can assume that there exists $T>0$ where $t_0\geq T$, such that

$u(x,t)>0,u(x,t-\tau)>0,u(x,t-\sigma)>0,u(x,t-\mu_j)>0,\ j=1,2,...,n$ for $(x,t)\in\Omega\times[t_0,\infty)$

From Lemma 2.1, we know that $z(t)$ is a positive solution of (2.1)-(2.3).

For $t\geq t_0,t\neq t_k,k=1,2,...,$ define

$w(t)=r(t)\dfrac{z^{'}(t)}{z(t)},t\geq t_0$ (2.11)

Then we have $w(t)>0,t\geq t_0,r(t)z^{'}(t)-w(t)z(t)=0$. We may assume that $z(t_0)=1$, thus we have that for $t\geq t_0$

$$z(t)=\exp\left(\int_{t_0}^{t}w(s)ds)\right), \tag{2.12}$$

$$z^{'}(t)=w(t)\exp\left(\int_{t_0}^{t}w(s)ds)\right), \tag{2.13}$$

$$z^{''}(t)=w^2(t)\exp\left(\int_{t_0}^{t}w(s)ds)\right)+w^1(t)\exp\left(\int_{t_0}^{t}w(s)ds)\right) \tag{2.14}$$

Substitute (2.12)-(2.14) into (2.1) and then we obtain,

$$r(t)w^2(t)\exp\left(\int_{t_0}^{t} w(s)ds)\right)+r(t)w^1(t)\exp\left(\int_{t_0}^{t} w(s)ds\right)+C(1-g(t))p(t)\exp\left(\int_{t_0}^{t-\sigma} w(s)ds)\right)$$
$$+(1-g(t))\sum_{j=1}^{n} C_j q_j(t)\exp\left(\int_{t_0}^{t-\mu_j} w(s)ds)\right)$$

Hence we have

$$r(t)w^2(t)+r(t)w^1(t)+C(1-g(t))p(t)\exp(-\int_{t-\sigma}^{t} w(s)ds)$$
$$+(1-g(t))\sum_{j=1}^{n} C_j q_j(t)\exp(-\int_{t-\mu_j}^{t} w(s)ds)\leq 0.t\neq t_k$$

$$r(t)w^1(t)+C(1-g(t))p(t)\exp(-\int_{t-\sigma}^{t} w(s)ds)$$
$$+(1-g(t))\sum_{j=1}^{n} C_j q_j(t)\exp\left(-\int_{t-\mu_j}^{t} w(s)ds\right)\leq 0.t\neq t_k,$$

From above inequality and condition $b_k\leq a_k^*$ it is easy to see that the function $w(t)$ is nonincreasing for $t\geq t_k\geq\delta+t_{0.}$. Thus $w(t)\leq w(t_{0.})$ for $t\geq t_{0.}$ which implies that

$$r(t)w^1(t)+C(1-g(t)p(t)\exp(-\delta w(t_{0.}))\sum_{j=1}^{n} C_j q_j(t)\leq 0, t\neq t_k.$$

From (2.2)-(2.3), we obtain

$$w(t_k^+)=r(t_k^+)\frac{z^{'}(t_k^+)}{z(t_k^+)}\leq r(t_k^+)\frac{a_k z^{'}(t_k^+)}{a_k^* z(t_k^+)}=r(t_k)\frac{a_k}{b_k^*}w(t_k),$$

and

$$r(t)w^1(t)\leq -C(1-g(t))\exp(-\delta w(t_{0.}))p(t)-(1-g(t))\exp(-\delta w(t_{0.}))\sum_{j=1}^{n} C_j q_j(t)\leq 0, t\neq t_k.$$

$$w(t_k^+)\leq r(t_k)\frac{b_k}{a_k^*}w(t_k), k=1,2,...$$

Let

$$-\psi(t)=\frac{\exp(-\delta w(t_{0.}))(1-g(t))\left[-Cp(t)-\sum_{j=1}^{n} C_j q_j(t)\right]}{r(t)}$$

Then according to Lemma 2.2, we have

$$w(t) \le w(t_{0.}) \prod_{t_{0.}<t_{k.}<t} r(t_k)\frac{b_k}{a_k^*} + \int_{t_{0.}}^{t} \prod_{s<t_{k.}<t} r(t_k)\frac{b_k}{a_k^*}\psi(s)ds$$

$$= \prod_{t_{0.}<t_{k.}<t} \frac{b_k}{a_k^*}\left[w(t_{0.})r(t_k) - \int_{t_{0.}}^{t} \prod_{t_{0.}<t_{k.}<t} r(t_k)\frac{a_k}{b_k^*}\psi(s)ds \right] < 0.$$

Since $w(t) \ge 0,$ the last inequality contradicts condition (2.10). This completes the proof.

3.OSCILLATION PROPERTIES OF THE PROBLEM (1.1) AND (1.3)

Next we consider the problem (1.1) and (1.3). To prove our main result we need the following lemma.

Lemma 3.1. Let $u(x,t) \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ be a positive solution of the problem (1.1),(1.3) in the domain G , then the function $z(t)$ satisfies the impulsive differential inequality

$$r(t)z^{"}(t) + C(1-g(t))p(t)z(t-\sigma)(1-g(t))\sum_{j=1}^{n} C_j q_j(t)z(t-\mu_j) \le 0, t \ne t_k \tag{3.1}$$

$$a_k^* \le \frac{z(t_k^+)}{z(t_k)} \le a_k \tag{3.2}$$

$$b_k^* \le \frac{z^{'}(t_k^+)}{z^{'}(t_k)} \le b_k, t = t_k, k = 1,2,3,... \tag{3.3}$$

Proof. Let $u(x,t)$ be a positive solution of the problem (1.1),(1.3) in G. Without loss of generality, we may assume that there exists a $T > 0$ where $t_0 > T$ such that

$$u(x,t) > 0, u(x,t-\tau) > 0, u(x,t-\sigma) > 0, u(x,t-\mu_j) > 0,\ j = 1,2,3,...,n.$$

For $t \ge t_0, t \ne t_k, k = 1,2,...,$ integrating (1.1) with respect to x , over Ω yields

$$\begin{aligned} &\frac{d}{dt}(r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx + g(t)\int_\Omega u\left(x,t-\tau)dx\right)\right) = a(t)\int_\Omega \Delta u(x,t)dx + \int_\Omega p(x,t) f\left(u(x,t-\tau)\right)dx \\ &-\sum_{j=1}^{n}\int_\Omega q_j(x,t) f_j\left(u(x,t-\mu_j)\right)dx + \int_\Omega F(x,t)dx. \end{aligned} \tag{3.4}$$

By Green's formula and the boundary condition (1.3) we have

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial \gamma} dS \le 0 \tag{3.5}$$

Where dS is the surface element on $\partial\Omega$ .The rest of the proof is similar to the Lemma 2.1, we omit it.

Using the above lemma, we prove the following oscillation result.

Theorem 3.1. If condition (2.9) and (2.10) hold, then each solution of (1.1),(1.3) oscillates in G

Proof. The proof is similar to Theorem 2.1 and hence the details are omitted.

4. EXAMPLES

In this section, we present an example to illustrate the main results.

Example 4.1. Consider the impulsive differential equation

$$\begin{aligned}&\frac{\partial}{\partial t}\left((t+\pi)^2\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{t+\pi}u\left(x,t-\frac{\pi}{2}\right)\right)\right)=(t+\pi)^2\Delta u(x,t)-2(t+\pi)u\left(x,t-\frac{5\pi}{2}\right)\\&-(t+\pi)u\left(x,t-\frac{9\pi}{2}\right),t>1,t\neq t_k,k=1,2,...\\&u(x,t_k^+)=\frac{k+1}{k}u(x,t_k)\\&u_t(x,t_k^+)=\frac{k+1}{k}u_t(x,t_k),k=1,2,...\end{aligned}\tag{4.1}$$

and the boundary condition

$$u(0,t)=u(\pi,t)=0,t>1,t\neq t_k,k=1,2,...\tag{4.2}$$

Here

$$\Omega=(0,\pi),a_k=a_k^*=\frac{k+1}{k},b_k=b_k^*=1,k=1,2,...r(t)=(t+\pi)^2,g(t)=\frac{1}{t+\pi},$$
$$a(t)=(t+\pi)^2,p(t)=2(t+\pi),q_1(t)=(t+\pi),\tau=\frac{\pi}{2},\sigma=\frac{5\pi}{2},\mu_1=\frac{9\pi}{2},f(u)=u,f_1(u)=u,$$

and taking $t_0=1,t_k=2^k,k=1,2,...$ Moreover

$$\begin{aligned}&\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{b_k^*}{a_k}ds=\int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds\\&=\int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_2^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_3^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds+...\\&=1+\frac{1}{2}\times2+\frac{1}{2}\times\frac{2}{3}\times2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times2^3+...\\&=\sum_{n=0}^{+\infty}\frac{2^n}{n+1}=+\infty\end{aligned}$$

so (2.9) holds. We take $\lambda=1,C=C_1=1,\delta=\max\{\tau,\sigma,\mu_1\}=\frac{9\pi}{2},w(t_0)=\frac{1}{t+\pi},$ then

$$\psi(t) = \frac{\exp\left\{-\frac{9\pi}{2}\times\frac{1}{t+\pi}\right\}\times\left\{1-\frac{1}{t+\pi}\right\}\times\left(2(t+\pi)+(t+\pi)\right)}{(t+\pi)}$$
$$= \frac{\exp\left\{-\frac{9\pi}{2}\times\frac{1}{t+\pi}\right\}\times 3\left\{1-\frac{1}{t+\pi}\right\}}{(t+\pi)}$$

Thus

$$\lim_{t\to+\infty}\int_1^t \prod_{1<t_k<s} \frac{a_k^*}{b_k} r(t_k)\psi(s)ds = \lim_{t\to+\infty}\int_1^t \prod_{1<t_k<s} \frac{k+1}{k}(2^k+\pi)^2 \times \frac{\exp\left\{-\frac{9\pi}{2}\times\frac{1}{t+\pi}\right\}\times 3\left\{1-\frac{1}{t+\pi}\right\}}{(s+\pi)} ds$$
$$> \int_1^t \frac{ds}{s+\pi} \to \infty, t\to\infty$$

Hence (2.10) holds. Therefore all conditions of Theorem 3.1 are satisfied. Hence every solution of the problem (4.1), (4.2) oscillates in $(0,\pi)\times[0,\infty)$. In fact $u(x,t)=\sin x\cos t$ is one such solution of the problem (4.1) and (4.2).

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