DERIVING SHAPE FUNCTIONS FOR CUBIC 12-NODED SERENDIPIDTY FAMILY ELMENT AND VERIFIED

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Abstract: In this paper, I derived shape functions for 12-noded cubicserendipity family element by using natural Co-ordinate system and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica9 Software [2].

Keywords: Cubic serendipity element, Natural Co-ordinate system, Shape functions.

1. INTRODUCTION

The approximating functions are defined in terms of field variables of specified points called nodes or nodal points. Thus in the finite element analysis the unknowns are the field variables of the nodal points. Once these are found the field variables at any point can be found by using interpolation functions/shape functions.

2. GEOMETRICAL DESCREPTION

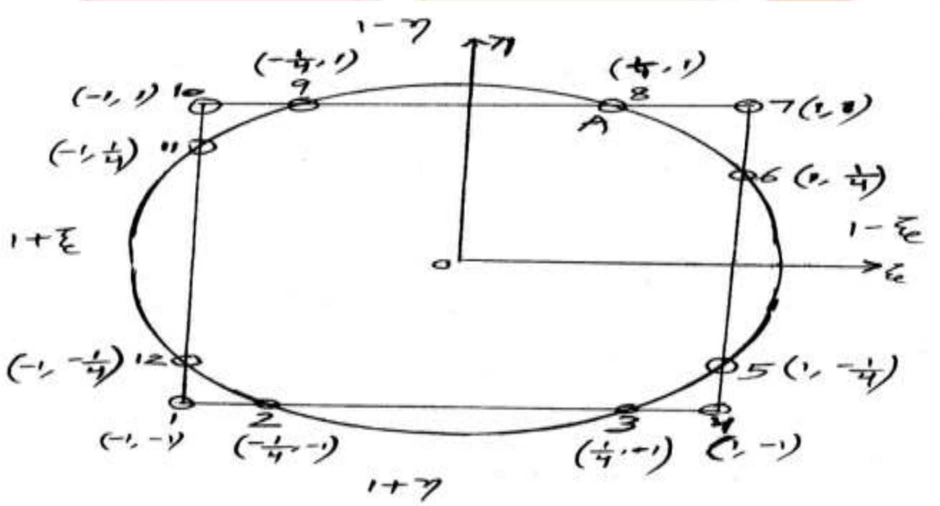

FIGURE.1 CUBICSERENDIPITY FAMILY ELEMENT

For computational purpose within the rectangle circle is inscribed. In this Rectangle 12 nodes are taken namely nodes 1,2,3,4,5,6,17,8,9,10,11,12. Whenever circle is inscribed in this rectangle, circle is coincided at nodes 2,3,5,6,8,9,11,12. If we want to analyze heat and mass transfer without taking corner points in that particular geometry then we use this type of geometry shown in figure.1.

3. DERIVING SHAPE FUNCTIONS FOR 12-NODEDCUBIC SERENDIPITY FAMILY ELEMENT

Typical element is shown in Figure.1.
Shape functions for Corner Nodes
$N_1 = 0$ is satisfied for nodes 4,5,6,7 if 1-ξ=0.
$N_1 = 0$ is satisfied for nodes 7,8,9,10 if 1-η=0.
The points 2,3,5,6,8,9,11,12 lie on the circle shown in Figure.1.

The radius of this circle=OA.

x_1, y_1 x_2, y_2

Since O=(0,0) and A=$(\frac{1}{4}, 1)$

Distance between two points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$OA = \sqrt{(\frac{1}{4} - 0)^2 + (1-0)^2} \Rightarrow OA = \sqrt{(\frac{1}{4})^2 + (1)^2} \Rightarrow OA = \sqrt{\frac{1}{16} + 1} \Rightarrow OA = \sqrt{\frac{1+16}{16}}$$

$$\Rightarrow OA = \sqrt{\frac{17}{16}} \Rightarrow OA = radius(r) = \sqrt{\frac{17}{16}}$$

$\therefore$ *The* equation of the circle is $x^2 + y^2 = r^2$

$$\xi^2 + \eta^2 = \left(\sqrt{\frac{17}{16}}\right)^2 \Rightarrow \xi^2 + \eta^2 = \frac{17}{16}$$

$$\xi^2 + \eta^2 - \frac{17}{16} = 0 \qquad (1)$$

eq(1) satisfies $N_1 = 0$ for nodes 2 to 12
Verification at node 2.

At Node 2 $\xi = -\frac{1}{4}, \eta = -1$

$$(1) \Rightarrow \left(-\frac{1}{4}\right)^2 + (-1)^2 - \frac{17}{16} = 0 \Rightarrow \frac{1}{16} + 1 - \frac{17}{16} = 0$$

$0 = 0$
eq(1) is verified at node 2.

For Corner node 1

$$Let\ N_1 = C(1-\xi)(1-\eta)(\xi^2+\eta^2-\frac{17}{16}) \qquad (2)$$

,satisfies $N_1 = 0$ for all nodes except for node 1.

For Node 1, $N_1 = 1$, $\xi = -1, \eta = -1$

$$1 = C(1-(-1))(1-(-1))((-1)^2+(-1)^2-\frac{17}{16}) \Rightarrow 1 = C(1+1)(1+1)\left(1+1-\frac{17}{16}\right)$$

$$1 = C(2)(2)\left(2-\frac{17}{16}\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(2) \Rightarrow N_1 = \frac{4}{15}(1-\xi)(1-\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (3)$$

For Corner node 4

$$Let\ N_4 = C(1+\xi)(1-\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (4)$$

,satifies $N_4 = 0$ for all nodes except for node 4.

For Node 4, $N_4 = 1$, $\xi=1$, $\eta = -1$.

$$(4) \Rightarrow 1 = C(1+1)(1-(-1))((1)^2+(-1)^2-\frac{17}{16}) \Rightarrow 1 = C(2)(1+1)\left(1+1-\frac{17}{16}\right)$$

$$1 = C(2)(2)\left(2-\frac{17}{16}\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(4) \Rightarrow N_4 = \frac{4}{15}(1+\xi)(1-\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (5)$$

For Corner node 7

$$N_7 = C(1+\xi)(1+\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (6)$$

satisfies $N_7 = 0$ for all nodes except for node 7.

For Node 7, $N_7 = 1$, $\xi=1$, $\eta=1$

$$(6) \Rightarrow 1 = C(1+1)(1+1)((1)^2+(1)^2-\frac{17}{16}) \Rightarrow 1 = C(2)(2)\left(1+1-\frac{17}{16}\right)$$

$$1 = C(4)\left(2-\frac{17}{16}\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(6) \Rightarrow N_7 = \frac{4}{15}(1+\xi)(1+\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (7)$$

For Corner node 10

$$Let\ N_{10} = C(1-\xi)(1+\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (8)$$

satisfies $N_{10} = 0$ for all nodes except for node 10.

For Node 10, $N_{10} = 1$, $\xi=-1$, $\eta=1$

$$(8) \Rightarrow 1 = C(1-(-1))(1+1)((-1)^2+(1)^2-\frac{17}{16}) \Rightarrow 1 = C(1+1)(2)\left(1+1-\frac{17}{16}\right)$$

$$1 = C(2)(2)\left(2-\frac{17}{16}\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(8) \Rightarrow N_{10} = \frac{4}{15}(1-\xi)(1+\eta)\left(\xi^2+\eta^2-\frac{17}{16}\right) \qquad (9)$$

For midside node 2

$1-\xi = 0$ ensures $N_2 = 0$ at nodes 4,5,6,7. ,$1-\eta = 0$ ensures $N_2 = 0$ at nodes 7,8,9,10.

$1+\xi = 0$ ensures $N_2 = 0$ at nodes 10,11,12,1. ,$1-4\xi = 0$ ensures $N_2 = 0$ at node 3.

$$Let\ N_2 = C(1-\xi)(1-\eta)(1+\xi)(1-4\xi) \qquad (10)$$

At Node 2, $N_2 = 1$, $\xi = -\frac{1}{4}$, $\eta = -1$

$$(10) \Rightarrow 1 = C(1-(-\frac{1}{4}))(1-(-1))\left(1+\left(-\frac{1}{4}\right)\right)\left(1-4\left(-\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(1+\frac{1}{4}\right)(1+1)\left(1-\frac{1}{4}\right)(1+1)$$

$$1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(10) \Rightarrow N_2 = \frac{4}{15}(1-\xi)(1-\eta)(1+\xi)(1\text{-}4\xi) \qquad (11)$$

For midside node 3

$1-\xi = 0$ ensures $N_3 = 0$ at nodes 4,5,6,7. ,$1-\eta = 0$ ensures $N_3 = 0$ at nodes 7,8,9,10.

$1+\xi = 0$ ensures $N_3 = 0$ at nodes 10,11,12,1. ,$1+4\xi = 0$ ensures $N_3 = 0$ at node 2.

$$Let\ N_3 = C(1-\xi)(1+\xi)(1-\eta)(1+4\xi) \qquad (12)$$

At Node 3, $N_3 = 1$, $\xi = \frac{1}{4}$, $\eta = -1$

$$(12) \Rightarrow 1 = C(1-\frac{1}{4})(1+\frac{1}{4})(1-(-1))\left(1+4\left(\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(\frac{4-1}{4}\right)\left(\frac{4+1}{4}\right)(1+1)(1+1)$$

$$1 = C\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)(2)(2) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(12) \Rightarrow N_3 = \frac{4}{15}(1-\xi)(1+\xi)(1-\eta)(1\text{+}4\xi) \qquad (13)$$

For midside node 5

$1-\eta = 0$ ensures $N_5 = 0$ at nodes 7,8,9,10.($\therefore \eta$=1 in nodes 7,8,9,10).

$1+\xi = 0$ ensures $N_5 = 0$ at nodes 10,11,12,1. ,$1+\eta = 0$ ensures $N_5 = 0$ at nodes 1,2,3,4.

$1-4\eta = 0$ ensures $N_5 = 0$ at node 6. ($\because \eta = \frac{1}{4}$ in node 6)

$$Let\ N_5 = C\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1-4\eta\right) \qquad (14)$$

At Node 5, $N_5 = 1$, $\xi=1$, $\eta = -\frac{1}{4}$

$$(14) \Rightarrow 1 = C(1+1)(1-\left(-\frac{1}{4}\right))\left(1+\left(-\frac{1}{4}\right)\right)\left(1-4\left(-\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(2\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{4}\right)\left(1+1\right)$$

$$1 = C\left(2\right)\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)\left(2\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(14) \Rightarrow N_5 = \frac{4}{15}\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1\text{-}4\eta\right) \qquad (15)$$

For midside node 6

$1-\eta = 0$ ensures $N_6 = 0$ at nodes 7,8,9,10. , $1+\xi = 0$ ensures $N_6 = 0$ at nodes 10,11,12,1.

$1+\eta = 0$ ensures $N_6 = 0$ at nodes 1,2,3,4. , $1+4\eta = 0$ ensures $N_6 = 0$ at node 5.

$$Let\ N_6 = C\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1+4\eta\right) \qquad (16)$$

At Node 6, $N_6 = 1$, $\xi=1$, $\eta = \frac{1}{4}$

$$(16) \Rightarrow 1 = C(1+1)(1-\frac{1}{4})\left(1+\frac{1}{4}\right)\left(1+4\left(\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(2\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(2\right)$$

$$1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(16) \Rightarrow N_6 = \frac{4}{15}\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1\text{+}4\eta\right) \qquad (17)$$

For midside node 8

$1+\xi = 0$ ensures $N_8 = 0$ at nodes 10,11,12,1. , $1+\eta = 0$ ensures $N_8 = 0$ at nodes 1,2,3,4.

$1-\xi = 0$ ensures $N_8 = 0$ at nodes 4,5,6,7. , $1+4\xi = 0$ ensures $N_8 = 0$ at node 9.

$$Let\ N_8 = C\left(1+\xi\right)\left(1-\xi\right)\left(1+\eta\right)\left(1+4\xi\right) \qquad (18)$$

At Node 8, $N_8 = 1$, $\xi=\frac{1}{4}$, $\eta = 1$

$$(18) \Rightarrow 1 = C(1+\frac{1}{4})(1-\frac{1}{4})\left(1+1\right)\left(1+4\left(\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)\left(2\right)\left(2\right)$$

$$1 = C\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)\left(2\right)\left(2\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(18) \Rightarrow N_8 = \frac{4}{15}\left(1+\xi\right)\left(1-\xi\right)\left(1+\eta\right)\left(1\text{+}4\xi\right) \qquad (19)$$

For midside node 9

$1+\xi = 0$ ensures $N_9 = 0$ at nodes 10,11,12,1. , $1+\eta = 0$ ensures $N_9 = 0$ at nodes 1,2,3,4.

$1-\xi = 0$ ensures $N_9 = 0$ at nodes 4,5,6,7. , $1-4\xi = 0$ ensures $N_9 = 0$ at node 8.

$$Let\ N_9 = C\left(1+\xi\right)\left(1-\xi\right)\left(1+\eta\right)\left(1-4\xi\right) \quad (20)$$

At Node 9, $N_9 = 1,\ \xi = -\frac{1}{4},\ \eta = 1$

$$(20) \Rightarrow 1 = C(1+\left(-\frac{1}{4}\right))(1-\left(-\frac{1}{4}\right))\left(1+1\right)\left(1-4\left(-\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)(2)(1+1)$$

$$\Rightarrow 1 = C\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(2\right)\left(2\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(20) \Rightarrow N_9 = \frac{4}{15}\left(1+\xi\right)\left(1-\xi\right)\left(1+\eta\right)\left(1\text{-}4\xi\right) \quad (21)$$

$$Let\ N_{11} = C\left(1-\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1+4\eta\right) \quad (22)$$

For midside node 11

$1+\eta = 0$ ensures $N_{11} = 0$ at nodes 1,2,3,4. , $1-\xi = 0$ ensures $N_{11} = 0$ at nodes 4,5,6,7.

$1-\eta = 0$ ensures $N_{11} = 0$ at nodes 7,8,9,10. , $1+4\eta = 0$ ensures $N_{11} = 0$ at node 12.

At Node 11, $N_{11} = 1,\ \xi = -1,\ \eta = \frac{1}{4}$

$$(22) \Rightarrow 1 = C(1-(-1))(1-\frac{1}{4})\left(1+\frac{1}{4}\right)\left(1+4\left(\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(1+1\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(1+1\right)$$

$$1 = C\left(2\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(2\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(22) \Rightarrow N_{11} = \frac{4}{15}\left(1-\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1\text{+}4\eta\right) \quad (23)$$

For midside node 12

$1+\eta = 0$ ensures $N_{12} = 0$ at nodes 1,2,3,4. , $1-\xi = 0$ ensures $N_{12} = 0$ at nodes 1,5,6,7.

$1-\eta = 0$ ensures $N_{12} = 0$ at nodes 7,8,9,10. , $1-4\eta = 0$ ensures $N_{12} = 0$ at node 11.

$$Let\ N_{12} = C\left(1-\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1-4\eta\right) \quad (24)$$

At Node 12, $N_{12} = 1,\ \xi = -1,\ \eta = -\frac{1}{4}$

$$(24) \Rightarrow 1 = C(1-(-1))(1-\left(-\frac{1}{4}\right))\left(1+\left(-\frac{1}{4}\right)\right)\left(1-4\left(-\frac{1}{4}\right)\right) \Rightarrow 1 = C\left(1+1\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{4}\right)\left(1+1\right)$$

$$1 = C\left(2\right)\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)\left(2\right) \Rightarrow 1 = C\left(\frac{15}{4}\right) \Rightarrow C = \frac{4}{15}$$

$$(24) \Rightarrow N_{12} = \frac{4}{15}\left(1-\xi\right)\left(1-\eta\right)\left(1+\eta\right)\left(1\text{-}4\eta\right) \quad (25)$$

$N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}$ are shape functions

4. VERIFICATION

(I) 1^{st} Condition

Sum of all the shape functions is equal to one

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 + N_9 + N_{10} + N_{11} + N_{12} = (3) + (10) + (13) + (5) +$$
$$+ (17) + (19) + (21) + (9) + (23) + (25)$$

Output

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 + N_9 + N_{10} + N_{11} + N_{12} = 1$$

II^{nd} Condition

Each shape function has a value of one at its own node and zero at the other nodes.

(II) 2nd Conditon

Each shape function has a value of one at its own node and zero at the other nodes

(i) *At* Node 1 (-1,-1) ξ:=-1 η:=-1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	1	0	0	0	0	0	0	0	0	0	0	0

(ii) *At* Node 2 $(-\frac{1}{4},-1)$ $\xi:=-\frac{1}{4}$ η:=-1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	1	0	0	0	0	0	0	0	0	0	0

(iii) *At* Node 3 $(\frac{1}{4},-1)$ $\xi:=\frac{1}{4}$ η:= -1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	1	0	0	0	0	0	0	0	0	0

(iv) *At* Node 4(1,-1) ξ:=1 η:= -1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	1	0	0	0	0	0	0	0	0

(v) *At* Node 5$(1,-\frac{1}{4})$ ξ:= 1 $\eta:= -\frac{1}{4}$

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	0	1	0	0	0	0	0	0	0

(vi) *At* Node 6$(1,\frac{1}{4})$ ξ:= 1 $\eta:= \frac{1}{4}$

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	0	0	1	0	0	0	0	0	0

(vii) *At* Node 7(1,1) ξ:= 1 η:= 1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	0	0	0	1	0	0	0	0	0

$(viii)$ *At* Node 8$(\frac{1}{4},1)$ $\xi:= \frac{1}{4}$ η:= 1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	0	0	0	0	1	0	0	0	0

(ix) *At* Node 9$(-\frac{1}{4},1)$ $\xi:= -\frac{1}{4}$ η:= 1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	0	0	0	0	0	1	0	0	0

(x) *At* Node 10(-1,1) ξ:= -1 η:= 1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
Output	0	0	0	0	0	0	0	0	0	1	0	0

(xi) *At* Node 11$(-1,\frac{1}{4})$ ξ:= -1 $\eta:= \frac{1}{4}$

N_1 N_2 N_3 N_4 N_5 N_6 N_7 N_8 N_9 N_{10} N_{11} N_{12} *Output* 0 0 0 0 0 0 0 0 0 0 1 0

(*xii*) *At* Node 12$(-1,-\frac{1}{4})$ $\xi := -1$ $\eta := -\frac{1}{4}$

N_1 N_2 N_3 N_4 N_5 N_6 N_7 N_8 N_9 N_{10} N_{11} N_{12} *Output* 0 0 0 0 0 0 0 0 0 0 0 1

5. CONCLUSIONS

1. Derived Shape functions for 12-noded rectangular cubic serendipity element
2. verified sum of all the shape functions is equal to one
3. Verified each shape function has a value of one at its own node and zero at the other nodes.

References

[1]. S.S. Bhavikatti, Finite Element Analysis, New AgeInternational (P) Limited, Publishers, 2nd Edition, 2010.
[2]. Mathematica9 Software, Wolfram Research, Version number 9.0.0.0, 1988-2012.