Fixed Point Results Using Common (E.A.) Property

Dr. Maheshwari P.G.
Department of Mathematics
Government First Grade College, Vijayanagar, Bangalore

Abstract: The main purpose of this paper is to use the notion of absorbing maps, which are neither a subclass of compatible maps nor a subclass of noncompatible maps, to establish unique common fixed point theorems under unified and extended strict contractive condition with weaker form of reciprocal continuity. These results extend and generalize the many recent results in the literature.

Keywords: , Common (E.A.) property, absorbing maps, Coincidence point, fixed point, g-reciprocal continuity.

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I. Introduction and Preliminaries

It is well known that in metric fixed point theory, weak compatibility and all its weaker versions requires commutativity at its coincidence points for the existence of common fixed points of contractive maps. Gopal et al. in [6], have tried to relax this commutativity condition by introducing a new notion known as absorbing maps, which are neither a class of compatible maps nor a subclass of noncompatible maps, as they do not enforce commutativity at its coincidence points unlike weakly compatible maps.

The problem "whether there exists a contractive definition which is strong enough to generate a fixed point, but which does not force the map to be continuous" was reiterated by Rhoades and has remained open for more than a decade. This problem was settled in the affirmative by Pant [2], by introducing the notion of reciprocal continuity, which is mainly applicable to the setting of compatible mappings. Many works under the analogue of reciprocal continuity have come through from past few years (see [8]-[11] and the references therein). Later, Pant et al. [4], have introduced this notion of g-reciprocal continuity, a new analogue of reciprocal continuity which ensures the existence of common fixed point under strict contractive conditions without assuming any stronger conditions on the space. Following this work, R.U. Joshi et al. [5] have proved a fixed point theorem for a generalized strict contractive condition, wherein the mappings involved are assumed to be g-compatible which enforce commutativity at its coincidence points.

In the present paper, we use the notion of absorbing maps [6], which are neither a subclass of compatible maps nor a subclass of noncompatible maps, as they do not enforce commutativity at its coincidence points. Further these results are the extension and generalization of the result of [5], [14] and many more results in the literature.

Before proceeding to further, we recollect some basic definitions which are needed in our main results.

Definition 1.1:[1] Two self maps f and g of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ for some t in X.}$$

Thus the mappings f and g will be non compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent.

Definition 1.2: [3] Two self mappings f and g of a metric space (X, d) are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 1.3:[4] Two self mappings f and g of a metric space (X, d) are called g- reciprocally continuous if and only if $\lim_{n\to\infty} ffx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 1.4: Two self maps f and g of a metric space (X, d) are called g-compatible if $\lim_{n\to\infty} d(ffx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 1.5: [6] A pair of self mappings *(f,g)* of a metric space *(X, d)* is called g- absorbing if there exists some real number $R > 0$ such that $d(gx, gfx) \le Rd(fx, gx)$ for all x in X.

Example 1.6. Let $X = [0, 1]$, d be the usual matric on X. Define $f, g : X \to X$ by $fx = 1$ if $x \neq 1$, $f1 = 0$ and $gx = 1$ for all x. Then one can easily verify that f is g-absorbing but it is not g-compatible.

In 2002, M. Aamri and D. El Moutawakil [7] introduced the property (E.A.), which is a true generalization of non compatible maps in metric spaces. For works on (E.A.) property refer [12-14].

Definition 1.7. [7] Let f and g be two self maps defined on a metric space (X,d). Then the pair (f,g) is said to satisfy the (E.A) property, if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ for some } t \in X.$$

Definition 1.8. Let (f,g) and (S,T) be two pairs of self mappings of a metric space (X,d). Then we say that (f, g) and (S,T) satisfies the common (E.A) property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \text{ for some } t \in X.$$

II. Main Result

We now state and prove our first main result.

Theorem 2.1. Let f, g and S be three self mappings of a metric space (X,d) satisfying

1. Common (E.A.) Property
2. $$d(fx,Sy) < max\left\{d(gx,gy),\frac{d(fx,gx)+d(Sy,gy)}{2},\frac{d(fx,gy)+d(Sy,gx)}{2}\right\} - min\left\{d(gx,gy),\frac{d(fx,gx)+d(Sy,gy)}{2},\frac{d(fx,gy)+d(Sy,gx)}{2}\right\} \quad \forall x \neq y$$
3. f and S are g-absorbing.

If one of the pair (f,g) or (S,g) is g-reciprocally continuous, then f, g and S have a unique common fixed point.

Proof. Given that *(f, g)* and *(S, g)* satisfies Common (E.A.) Property. Then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} gy_n = t \text{ for some } t \in X .$$

If *(f, g)* is g-reciprocally continuous, then $ffx_n \to ft$ and $gfx_n \to gt$ as $n \to \infty$.
Since f is g-absorbing, there exists $R_1 > 0$ such that

$$d(gx, gfx) \le R_1\, d(fx, gx) \quad \text{for all } x \in X.$$

Consider $d(gx_n, gfx_n) \le R_1 d(fx_n, gx_n)$. On letting $n \to \infty$ we obtain $gt = t$. Now we will prove that $ft = t$. Consider

$$d(ffx_n, Sy_n) < max\left\{d(gfx_n, gy_n), \frac{d(ffx_n, gfx_n) + d(Sy_n, gy_n)}{2}, \frac{d(ffx_n, gy_n) + d(Sy_n, gfx_n)}{2}\right\} - min\left\{d(gfx_n, gy_n), \frac{d(ffx_n, gfx_n) + d(Sy_n, gy_n)}{2}, \frac{d(ffx_n, gy_n) + d(Sy_n, gfx_n)}{2}\right\}$$

on letting $n \to \infty$ we get $d(ft,t) \leq \frac{d(ft,t)}{2}$, which gives $ft = t = gt$

Similarly consider

$$d(ffx_n, St) < max\left\{d(gfx_n, gt), \frac{d(ffx_n, gfx_n) + d(St, gt)}{2}, \frac{d(ffx_n, gt_n) + d(St, gfx_n)}{2}\right\} - min\left\{d(gfx_n, gt), \frac{d(ffx_n, gfx_n) + d(St, gt)}{2}, \frac{d(ffx_n, gt_n) + d(St, gfx_n)}{2}\right\}$$

on letting $n \to \infty$ we get $ft = St = gt = t$. Thus t is a common fixed point of f, g and S.

Next, suppose that (S, g) is g-reciprocally continuous, then $SSy_n \to St$ and $gSy_n \to gt$ as $n \to \infty$. Since S is g-absorbing, there exists $R_2 > 0$ such that

$$d(gx, gSx) \leq R_2 d(Sx, gx) \quad \text{for all } x \in X.$$

Consider $d(gy_n, gSy_n) \leq R_2 d(Sy_n, gy_n)$. On letting $n \to \infty$ we obtain $gt = t$. Now we will prove that $St = t$. Consider

$$d(fx_n, SSy_n) < max\left\{d(gx_n, gSy_n), \frac{d(fx_n, gx_n) + d(SSy_n, gSy_n)}{2}, \frac{d(fx_n, gSy_n) + d(SSy_n, gx_n)}{2}\right\} - min\left\{d(gx_n, gSy_n), \frac{d(fx_n, gx_n) + d(SSy_n, gSy_n)}{2}, \frac{d(fx_n, gSy_n) + d(SSy_n, gx_n)}{2}\right\}$$

on letting $n \to \infty$ we get $d(t, St) \leq \frac{d(t,St)}{2}$, which gives $St = t = gt$

Similarly consider

$$d(ft, SSy_n) < max\left\{d(gt, gSy_n), \frac{d(ft, gt) + d(SSy_n, gSy_n)}{2}, \frac{d(ft, gSy_n) + d(SSy_n, gt)}{2}\right\} - min\left\{d(gt, gSy_n), \frac{d(ft, gt) + d(SSy_n, gSy_n)}{2}, \frac{d(ft, gSy_n) + d(SSy_n, gt)}{2}\right\}$$

on letting $n \to \infty$ we get $ft = St = gt = t$. Thus t is a common fixed point of f, g and S.

Uniqueness: Let u and v be the two common fixed points of f, g and S. Then,

$$u = fu = gu = Su \qquad \text{and} \qquad v = fv = gv = Sv$$

Now we have to prove that $u = v$. Suppose that $u \neq v$. Then,

$$d(u, v) = d(fu, Sv) < \max\left\{d(gu, gv), \frac{d(fu,gu)+d(Sv,gv)}{2}, \frac{d(fu,gv)+d(Sv,gu)}{2}\right\} - \min\left\{d(gu, gv), \frac{d(fu,gu)+d(Sv,gv)}{2}, \frac{d(fu,gv)+d(Sv,gu)}{2}\right\}$$

which gives $d(u, v) < d(u, v)$, a contradiction. Hence $u = v$.

The above theorem is illustrated by the following example.

Example 2.2. Let $X = [2, 10]$ and d be the usual metric on X. Define $f, g, S : X \to X$ by

$$Sx = 2 \ \forall x, \qquad f(x) = \begin{cases} 2, & if\ x = 2\ or\ x > 5 \\ 4, & if\ 2 < x \leq 3 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & if\ x = 2 \\ 6, & if\ 2 < x \leq 5 \\ \frac{x+1}{3}, & if\ x > 5 \end{cases}$$

Now (f, g) and (S, g) are g-reciprocally continuous. To see this, choose two sequences $\{x_n\} =$

$\left\{5+\frac{1}{n}\right\}$ and $\{y_n\} = \left\{5+\frac{2}{n}\right\}$ for all n. Then $fx_n = f\left(5+\frac{1}{n}\right) \to 2$, $gx_n = g\left(5+\frac{1}{n}\right) \to 2$, $Sy_n = S\left(5+\frac{2}{n}\right) \to 2$ and $gy_n = g\left(5+\frac{2}{n}\right) \to 2$. Therefore $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} gy_n = 2$.

Thus *(f, g)* and *(S, g)* satisfies the common (E.A.) property and are g-reciprocally continuous. Also for all $x \in X$ we have

$d(gx, gfx) \leq R_1 d(fx, gx)$ and $d(gx, gSx) \leq R_2 d(Sx, gx)$ where $R_1, R_2 \geq 1$.

Therefore, f and S are g-absorbing. Further, f, g and S satisfy all the conditions of Theorem 2.1 and have a unique common fixed point at $x = 2$.

Corollary 2.3. Let f and g be g-reciprocally continuous self mappings of a metric space (X,d) satisfying

1. (E.A.) Property
2. $d(fx, fy) < \max\left\{d(gx, gy), \frac{d(fx,gx)+d(gy,gy)}{2}, \frac{d(fx,gy)+d(gy,gx)}{2}\right\} - \min\left\{d(gx, gy), \frac{d(fx,gx)+d(gy,gy)}{2}, \frac{d(fx,gy)+d(gy,gx)}{2}\right\} \quad \forall x \neq y$
3. f is g-absorbing.

Then f and g have a unique common fixed point.

Proof. Put $S = f$ in Theorem 2.1

Theorem 2.5. Let f, g, S and T be four self mappings of a metric space (X,d) satisfying

1. Common (E.A.) Property
2. $d(fx, Sy) < \max\left\{d(gx, Ty), \frac{d(fx,gx)+d(Sy,Ty)}{2}, \frac{d(fx,Ty)+d(Sy,gx)}{2}\right\} - \min\left\{d(gx, gy), \frac{d(fx,gx)+d(Sy,gy)}{2}, \frac{d(fx,gy)+d(Sy,gx)}{2}\right\} \quad \forall x \neq y$
3. f is g-absorbing and S are T-absorbing.

Then f, g, S and T have a unique common fixed point.

Proof. Given that *(f, g)* and *(S, g)* satisfies Common (E.A.) Property. Then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \text{ for some } t \in X.$$

Since f is g-absorbing and S is T-absorbing, there exists $R_1, R_2 > 0$ such that

$$d(gx, gfx) \leq R_1 d(fx, gx) \quad \text{and} \quad d(Tx, TSx) \leq R_2 d(Sx, Tx) \quad \text{for all } x \in X.$$

Now (f, g) and *(S, T)* are g-reciprocally continuous implies

$$ffx_n \to ft,\ gfx_n \to gt,\ SSy_n \to St \text{ and } TSy_n \to Tt \text{ as } n \to \infty.$$

Consider $d(gx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$. On letting $n \to \infty$ we obtain $gt = t$.
Similarly $d(Ty_n, TSy_n) \leq R_2 d(Sy_n, Ty_n)$ gives $Tt = t$.
Now we will prove that $ft = t$. Consider

$$d(ffx_n, Sy_n) < \max\left\{d(gfx_n, Ty_n), \frac{d(ffx_n,gfx_n)+d(Sy_n,Ty_n)}{2}, \frac{d(ffx_n,Ty_n)+d(Sy_n,gfx_n)}{2}\right\} - \min\left\{d(gfx_n, Ty_n), \frac{d(ffx_n,gfx_n)+d(Sy_n,Ty_n)}{2}, \frac{d(ffx_n,Ty_n)+d(Sy_n,gfx_n)}{2}\right\}$$

on letting $n \to \infty$ we get $d(ft, t) \leq \frac{d(ft,t)}{2}$, which gives $ft = t = gt$

Similarly consider

$$d(fx_n, SSy_n) < \max\left\{d(gx_n, TSy_n), \frac{d(fx_n, gx_n) + d(SSy_n, TSy_n)}{2}, \frac{d(fx_n, TSy_n) + d(SSy_n, gx_n)}{2}\right\} - \min\left\{d(gx_n, TSy_n), \frac{d(fx_n, gx_n) + d(SSy_n, TSy_n)}{2}, \frac{d(fx_n, TSy_n) + d(SSy_n, gx_n)}{2}\right\}$$

on letting $n \to \infty$ we get $d(t, St) \le \frac{d(t,St)}{2}$, which gives $St = t = gt$

Hence $ft = St = gt = Tt = t$. Thus t is a common fixed point of f, g, S and T.

Uniqueness: Let u and v be the two common fixed points of f, g, S and T . Then,

$$u = fu = gu = Su = Tu \quad \text{and} \quad v = fv = gv = Sv = Tv$$

Now we have to prove that $u = v$. Suppose that $u \neq v$. Then,

$$d(u, v) = d(fu, Sv) < \max\left\{d(gu, Tv), \frac{d(fu,gu)+d(Sv,Tv)}{2}, \frac{d(fu,Tv)+d(Sv,gu)}{2}\right\} - \min\left\{d(gu, Tv), \frac{d(fu,gu)+d(Sv,Tv)}{2}, \frac{d(fu,Tv)+d(Sv,gu)}{2}\right\}$$

which gives $d(u, v) < d(u, v)$, a contradiction. Hence $u = v$.

The above theorem is illustrated by the following example.

Example 2.6. Let $X = [1, 10]$ and d be the usual metric on X. Define $f, g, S, T : X \to X$ by

$$f(x) = \begin{cases} 3, & if\ x \le 3 \\ 4, & if\ x > 3 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 6 - x, & if\ x \le 3 \\ 1, & if\ x > 3 \end{cases}$$

$$g(x) = \begin{cases} 2, & if\ x < 3 \\ \frac{x+3}{2}, & if\ x \ge 3 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 9, & if\ x < 3 \\ \frac{2x+3}{3}, & if\ x \ge 3 \end{cases}$$

The mappings (f, g) and (S, T) are g-reciprocally continuous. To see this, choose two sequences $\{x_n\} = \left\{3 - \frac{1}{n}\right\}$ and $\{y_n\} = \left\{3 + \frac{1}{n}\right\}$ for all n. Then $fx_n = f\left(3 - \frac{1}{n}\right) \to 3$, $gx_n = g\left(3 - \frac{1}{n}\right) \to 3$, $Sy_n = S\left(3 + \frac{1}{n}\right) \to 3$ and $Ty_n = T\left(3 + \frac{1}{n}\right) \to 3$. Therefore $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \mathrm{S}x_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} \mathrm{T}y_n = 3$

Thus (f, g) and (S, g) satisfies the common (E.A.) property and are g-reciprocally continuous. Also for all $x \in X$ we have

$$d(gx, gfx) \le R_1 d(fx, gx) \quad \text{and} \quad d(gx, gSx) \le R_2 d(Sx, gx) \quad \text{where } R_1, R_2 \ge 1.$$

Therefore f is g-absorbing and S is T-absorbing. Further, f, g and S satisfy all the conditions of Theorem 2.3 and have a unique common fixed point at $x = 3$.

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