MATHEMATICAL MODELLING OF COTTAGE INDUSTRY BY DIMENSIONAL ANALYSIS AN APPROPRIATE APPROACH - A REVIEW.

G. V. Thakre[1], S.G.Patil[2],
1. Bapurao Deshmukh College Of Engineering Sevagram
2, IBSS college of Engineering Ghatkheda Amravati

ABSTRACT:

Dimensional Analysis can make a contribution to model formation when some of the measurements in the problem are of physical factors. The analysis constructs the set of independent dimensionless factors that should be used as the major variables. The Dimensional Analysis technique has been used in various applications in most experimentally based areas of physical sciences and engineering. The basic theme of Dimensional Analysis is the Buckingham – л Theorem. The Dimensional Analysis actually follows the fundamental units of relevant quantities. The number of dimensional product is the difference between number of variables and the rank of matrix.

Dimensional analysis is most useful when a mathematical model is not known. The first and crucial step of dimensional analysis is to define a suitably idealized representation of a phenomenon by listing the relevant variables, called the physical model. The second step is to learn the consequences of the physical model and the general principle that complete equations are independent of the choice of units. The calculation that follows yields a basis set of non-dimensional variables. The final step is to interpret the non-dimensional basis set in the light of observations or existing theory, and if necessary to modify the basis set to maximize its utility.

Keywords: Ambar charkha, Dimensional Analysis, Mathematical Modelling.

1. INTRODUCTION:

1.1 AMBAR CHARKHA:

Mahatma Gandhiji believed that Charkha as a tool can make people self-sufficient. Charkha is a unique device to spin yarn. Originally, designed as the “Ambar charkha” (meaning "sky wheel"), about 40 years ago, had increased the productivity of user spinners and enabled them to earn up to Rs. 50 per day. Ambar Charkha designed by Ekambarnath, a Gandhian worker from Tamil Nadu, following an appeal by Mahatma Gandhiji for a more productive version of the charkha. It may not look like the typical charkha made of wood and a wheel attached to it. But it is still a simple device and can be operated even by a child.[6]

There are two types of Ambar Charkha used at Gram Seva Mandal, Wardha, 6 spindle and 8 spindle. Machine i.e. charkha’s specifications are shown in tabulated form as follows:

Fig.1. Ambar Charkha

Sr. No.	SPECIFICATION	6 SPINDLE	8 SPINDLE
1.	Speed	40-45 rpm	30-35 rpm
2.	Handle Length	381 mm	400 mm
3.	Handle Diameter	0.5 cm	1 cm
4.	Weight of Machine	35 Kg	45 Kg
5.	Height of Machine	508 mm	660 mm
6.	Width of Machine	762 mm	914 mm

Table1.Specification of Ambar Charkha

1.2 DIMENSIONAL ANALYSIS:

Dimensional analysis is a widely applicable and sometimes very powerful technique. The first and crucial step of dimensional analysis is to define a suitably idealized representation of a phenomenon by listing the relevant variables, called the physical model. The second step is to learn the consequences of the physical model and the general principle that complete equations are independent of the choice of units. The calculation that follows yields a basis set of non-dimensional variables. The final step is to interpret the non-dimensional basis set in the light of observations or existing theory, and if necessary to modify the basis set to maximize its utility. One strategy is to non-dimesionalise the dependent variable by a scaling estimate. The remaining non-dimensional variables can then be formed in ways that define aspect ratios or that corresponds to the ratio of terms in a governing equation.[8]

Dimensional analysis is based on the fact that physically based phenomenon do not depend on the units chosen to describe their variables. It is one of the most effective tools for the analysis of industrial processes and often provides substantial insight with very little effort.[2]

This paper is an introduction to dimensional analysis that aims to make the method and the results as accessible as possible.

1.2.1The steps of dimensional analysis and Buckingham's л –theorem:

The premise of dimensional analysis is that the form of any physically significant equation must be such that the relationship between the actual physical quantities remains valid independent the magnitudes of the base units. Dimensional analysis derives the logical consequences of this premise.

Suppose we are interested in some particular physical quantity Q_0 that is a "dependent variable" in a well-defined physical process or event. By this we mean that, once all the quantities that define the particular process or event are specified, the value of Q_0 follows uniquely. [2]

Step 1: The independent variables:

The first and most important step in dimensional analysis is to identify a complete set of independent quantities $Q_2...Q_n$ that determine the value of Q_0,

$$Q_0 = f\,(Q_1, Q_2, Qn) \qquad (1.2.1)$$

A set $Q_{1..}$...Qn is complete if, once the values of the members are specified, no other quantity can affect the value of Q_0, and independent if the value of each member can be adjusted arbitrarily without affecting the value of any other member.

Starting with a correct set $Q_1.....Q_n$ is as important in dimensional analysis as it is in mathematical physics to start with the correct fundamental equations and boundary conditions.

The relationship expressed symbolically in equation (1.2.1) is the result of the physical laws that govern the phenomenon of interest. It is our premise that its form must be such that, once the values $Q_1...Q_n$ are specified, the equality holds regardless of the sizes of the base units in terms of which the quantities are measured. The steps that follow derive the consequences of this premise.[2]

Step 2: Dimensional considerations

Next we list the dimensions of the dependent variable Q_0 and the independent variables $Q_1...Q_n$. As we have discussed, the dimension of a quantity depends on the type of system of units, and we must specify at least the type the system of units before we do this. For example, if we use a system which is dealing with a purely mechanical problem, all quantities have dimensions of the form

$$[Q_i]=L^{l}{}_{i}M^{m}{}_{i}T^{\tau}{}_{i} \qquad (1.2.2)$$

Where, the exponent l_i, m_i and τ_i are dimensionless numbers that follow from each quantity's definition.We now pick from the complete set of physically independent variables $Q_1...Q_n$ a complete, dimensionally independent subset $Q_1...Q_k(k\leq n)$, and express the dimension of each of the remaining independent variables $Q_{k+1}...Q_n$ and the dependent variable Q_0 as a product of powers of $Q_1...Q_k$. All physical quantities have dimensions which can be expressed as products of powers of the set of base dimensions. Alternatively, it is possible to express the dimension of one quantity as a product of powers of the dimensions of other quantities which are not necessarily base quantities. A subset $Q_1...Q_k$ of the set $Q_1...Q_n$ is dimensionally independent if none of its members has a dimension that can be expressed in terms of the dimensions of the remaining members. And complete if the dimensions of all the remaining quantities $Q_{k+1}...Q_n$ of the full set can be expressed in terms of the dimensions of the subset $Q_1...Q_k$. Since equation (1.2.1) is dimensionally homogeneous, the dimension of the dependent variable Q_0 is also expressible in terms of the dimensions of $Q_1....Q_k$.[2]

The dimensionally independent subset $Q_1...Q_k$ is picked by trial and error. Its members may be picked in different ways (see section 1.2.3), but the number k of dimensionally independent quantities in the full set $Q_1...Q_n$ is unique to the set, and cannot exceed the number of base dimensions which appear in the dimensions the quantities in that set. For example, if the dimensions of $Q_1...Q_n$ involve only length, mass, and time, then $k\leq 3$.

Having chosen a complete, dimensionally independent subset $Q_1...Q_k$, we express the dimensions of Q_0 and the remaining quantities $Q_{k+1}...Q_n$ in terms of the dimensions of $Q_1...Q_k$. These will have the form

$$[Q_i]=[Q_1{}^{N}{}_{i1}Q_2{}^{N}{}_{i2}.........Q_k{}^{N}{}_{ik}] \qquad (1.2.3)$$

If $i>k$ or $i=0$. The exponents N_{ij} are dimensionless real numbers and can in most cases be found quickly by inspection (see section 1.2.2), although a formal algebraic method can be used.

The formal procedure can be illustrated with an example where length, mass and time are the only base quantities, in which case all dimensions have the form of equation (1.2.2). Let us take Q_1, Q_2, and Q_3 as the complete dimensionally independent subset. Equating the dimension given by equation (1.2.2) with that of equation (1.2.3), we obtain three equations

$$l_i=\sum_{j=1}^{3}N_{ij}l_j \qquad m_i=\sum_{j=1}^{3}N_{ij}m_j \qquad t_i=\sum_{j=1}^{3}N_{ij}t_j \qquad (1.2.4)$$

This can be solved for the three unknowns Ni1, Ni2, and Ni3.

Step 3: Dimensionless variables

We now define dimensionless forms of the n-k remaining independent variables by dividing each one with the product of powers of $Q_1...Q_k$which has the same dimension,

$$\pi_i=\frac{Q_{k+1}}{Q_1{}^{N}{}_{(k+1)1}Q_2{}^{N}{}_{(k+1)2}...Q_k{}^{N}{}_{(k+1)k}} \qquad (1.2.5)$$

Where,i=1, 2,...,, n-k, and a dimensionless form of the dependent variableQ_0,

$$\pi_o = \frac{Q_0}{Q_1{}^{N_{01}} Q_2{}^{N_{02}} \ldots Q_k{}^{N_{0k}}} \qquad (1.2.6)$$

Step 4: The end game and Buckingham's -theorem

An alternative form of equation (1.2.1) is

$$\pi_0 = f(Q_1, Q_2, \ldots, Q_k;\ \pi_1, \pi_2, \ldots, \pi_{n-k}) \qquad (1.2.7)$$

In which all quantities are dimensionless except $Q_1 \ldots Q_k$. The values of the dimensionless quantities are independent of the sizes of the base units. The values of $Q_1 \ldots Q_k$, on the other hand, do depend on base unit size. They cannot be put into dimensionless form since they are (by definition) dimensionally independent of each other. From the principle that any physically meaningful equation must be dimensionally homogeneous, that is, valid independent of the sizes of the base units, it follows that $Q_1 \ldots Q_k$must in fact be absent from equation (1.2.7), that is,

$$\pi = f(\pi_1, \pi_2, \ldots, \pi_{n-k}) \qquad (1.2.8)$$

This equation is the final result of the dimensional analysis, and contains Buckingham's -theorem:

When a complete relationship between dimensional physical quantities is expressed in dimensionless form, the number of independent quantities that appear in it is reduced from the original n to n-k, where k is the maximum number of the original n that are dimensionally independent. The theorem derives its name from Buckingham's use of the symbol for the dimensionless variables in his original 1914 paper. The π-theorem tells us that, because all complete physical equations must be dimensionally homogeneous, a restatement of any such equation in an appropriate dimensionless form will reduce the number of independent quantities in the problem by k. This can simplify the problem enormously, as will be evident from the example that follows.

The π−theorem itself merely tells us the number of dimensionless quantities that affect the value of a particular dimensionless dependent variable. It does not tell us the forms of the dimensionless variables. That has to be discovered in the third and fourth steps described above. Nor does the π-theorem, or for that matter dimensional analysis as such, say anything about the form of the functional relationship expressed by equation (1.2.7). That form has to be discovered by experimentation or by solving the problem theoretically.[2]

Example: Deformation of an elastic ball striking a wall:

Suppose we wish to investigate the deformation that occurs in elastic balls when they impact on a wall. We might be interested, for example, in finding out what determines the diameter d of the circular imprint left onthe wall after a freshly dyed ball has rebounded from it (figure 2).

Step 1: The independent variables:

The first step is to identify a complete set of independent quantities that determine the imprint radius d. We begin by specifying the problem more clearly. We agree to restrict our attention to (initially) spherical, homogeneous balls made of perfectly elastic material, to impacts at perpendicular to the wall, and to walls that are perfectly smooth and flatand so stiff and heavy that they do not deform or move during the impact process.

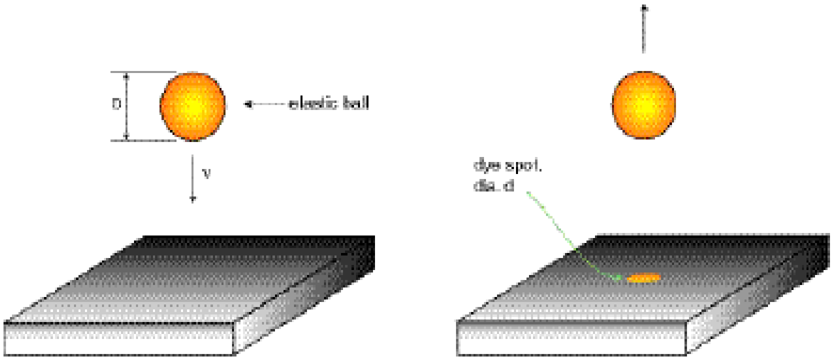

Fig.2. A freshly dyed elastic ball leaving imprint after impact with rigid wall.

The numerical value of a dependent variable like d will be depend on the values of all quantities that distinguish one impact event from another. Experience suggests that these should include at least the following: the ball's diameter D and velocity V just prior to contact (the initial conditions) and its mass m. The ball's intrinsic material properties will also play a role. Our theoretical understanding of solid mechanics tells us that the quasi-static response of a perfectly elastic material is characterized by two material properties, the modulus of elasticity E and Poisson's ratio, and that the inertial effects which inevitably come into play during collision and rebound will also depend on the material's density. The properties of the wall are irrelevant if it is indeed perfectly rigid, as we assumed. We know, however, by thinking of how the problem would have to be set up as a theoretical one that the answer for the numerical value d will also depend on the values of all universal constants that appear in the physical laws that control the ball's impact dynamics. In this case the process is governed by Newton's law of motion and the law of mass conservation. We know that Newton's law has the form F=ma and contains no universalconstants. Nor are there any physical constants in the law of mass conservation.

We seem to arrive at the conclusion that d depends on six quantities: D, V, m, E, γ, ρ. This is a complete set, as required, but not an independent set: once the ball's mass and diameter are specified, its density follows.We must therefore exclude either the density or the mass. (Other quantities like V^2, $DE^{1/2}$, etc, all involving quantities that affect the value of d, are excluded for the same reason: they are not independent of the quantities already included.) We conclude that the following relationship expresses the impact diameter in terms of a complete set of independent variables:

$$d = d(V, \rho, D, E, \gamma) \tag{1.2.9}$$

Note that the choice of a complete, independent set for a specified problem is not unique except for the number n of its members (n=5 in this case). One could just as well have chosen V^2, ρ, D, E, γ and say, or V, m, D, E, γ —see section 1.2.3. It should also be noted that further assumptions have been taken for granted in equation (1.2.9). We have presumed, for example, that the ball's motion is unaffected by the properties of the fluid through which it approaches the wall, and that gravitational effects play a negligible role.

Step 2: Dimensional considerations

In the type of system of units we have adopted in step 1, the dimensions of the quantities in equation (1.2.9) are:

Independent: $[V]=Lt^{-1}$, $[\rho]=ML^{-3}$, $[D]=L$, $[E]=ML^{-1}t^{-2}$, $[\gamma]=1$
Dependent: $[d]=L$ (1.2.10)

Inspection of the above shows that the three quantities V, and D, for example, comprise a complete, dimensionally independent subset of the five independent variables. The dimension of any one of these three cannot be made up of the dimensions of the other two. The dimensions of the remaining independent variables E and the dependent variable d can, however, be made up of those of V, and D as follows:

Independent: $[E]= ML^{-1}t^{-2}= (ML^{-3})(Lt)^2=[\rho\ V^2]$, $[\gamma\ ]=1$

Dependent: $[d]= L=[D]$ (1.2.11)

We have written down these results very simply by inspection. Accomplished practitioners seldom use the formal algebraic method of section (1.2.1). Note again that the dimension of a dimensionless quantity like is unity, the factor by which dimensionless numbers change when the sizes of the base units are changed.

Step 3: Dimensionless similarity parameters

We non-dimensionalise the remaining independent variables E and γ,the dependent variable d by dividing them by ρV^2, D and unity, respectively, as suggested by equation (1.2.11):

Independent: $\pi_1 = \frac{ED^3}{mV^2}$, $\pi_2 = \gamma$

Dependent: $\pi_0 = \frac{d}{D}$ (1.2.12)

Step 4: The end game

Using the logic that led to Buckingham's π−theorem, we now conclude that

$$\pi_0 = f(\pi_1, \pi_2) \qquad \text{Or} \qquad \frac{d}{D} = f\left(\frac{E}{\rho V^2}, \gamma\right) \tag{1.2.13}$$

The number of independent variables has been reduced from the original n=5 that define the problem to n-k =2.

1.2.2On the utility of dimensional analysis, some difficulties arise in its application:

A) Similarity:

Dimensional analysis provides a similarity law for the phenomenon under consideration. Similarity in this context implies certain equivalence between two physical phenomena that are actually different. The collision of two different elastic spheres 1 and 2 with a rigid wall, each with its own values of V, ρ, D, E and γ may appear to be quite different. However, under particular conditions where the parameters of the two events are such that 1 and 2 have the same values, that is, where

$$\frac{E_2}{\rho_2 {V_2}^2} = \frac{E_1}{\rho_1 {V_1}^2}; \qquad \gamma_2 = \gamma_1 \tag{1.2.14}$$

Equation (1.2.13) informs us that 0 has the same value in both cases, that is,

$$\frac{d_2}{D_2} = \frac{d_1}{D_1} \tag{1.2.15}$$

When the relationships in equation (1.2.14) apply, the two dynamic events are similar in the sense of equation (1.2.15).[2]

B) Out-of-scale modelling:

Scale modelling deals with the following question: If we want to learn something about the performance of a full-scale system 1 by testing a geometrically similar small-scale system model 2 (or vice versa, if the system of interest inaccessibly small), at what conditions should we test the model, and how should we obtain the full-scale performance from measurements at the small scale? Dimensional analysis provides the answer.

Suppose we need to know the deformation diameter of a huge, soft rubber ball with a diameter D_1 of 5 meters and properties E_1, ρ_1, and γ_1, as it hits the pavement with a speed V_2 of 10 m/s, but are unable to compute it from basic principles. In that case, we need only perform one small-scale test with a model 2 of diameter D_2, selecting its properties and test conditions such that equations (1.2.14) are satisfied, and measure its imprint diameter d_2. The full-scale value d_1 of the big ball's imprint diameter at its "design conditions" can then be obtained from equation (1.2.15).[2]

C) An incomplete set of independent quantities may destroy the analysis

Assuming competence on the part of the analyst, the correctness of the dimensional analysis will depend entirely on whether a complete set of independent quantities $Q_1...Q_n$ is in fact properly identified in step 1. Any complete set will yield correct results. If, however, the analysis is based on a set which omits even one independent quantity that affects the value ofQ_0, dimensional analysis will give erroneous results.

Suppose that in our example we had omitted the sphere's modulus of elasticity E in equation (1.2.9). Instead of equation (1.2.13), we would then have obtained the absurd result

$$\frac{d}{D} = f(\gamma) \tag{1.2.16}$$

Which implies that the maximum deformation depends on the ball's Poisson ratio, but is independent of its elasticity, mass and approach velocity! This single error of omission is clearly fatal to the analysis.[2]

D) Superfluous independent quantities complicate the result unnecessarily:

Errors on the side of excess have a less traumatic effect. Over specification of independent variables does not destroy the analysis, but robs it of its power. For every superfluous independent quantity included in the set, there will be in the final dimensionless relationship a superfluous dimensionless similarity parameter.

Suppose we argue that the ball's deformation upon impact will in general also depend on the local gravitational acceleration g (which we assume to be in the direction into the wall on which the impact occurs). This would change equation (1.2.13) to

$$\frac{d}{D} = f\left(\frac{E}{\rho V^2}, \gamma, \frac{gD}{V^2}\right) \tag{1.2.17}$$

Where,$gD/V^2$is a dimensionless gravity. Under conditions where the deformation is in fact insensitive to gravity, as we implicitly assumed earlier, equation (1.2.17) is "wrong" only in the sense that it suggests a dependence on g that is not noticeably there, and thus unnecessarily complicates our thinking. If by experimentation or computation we eventually discover that there exists a broad range of conditions where the similarity parameter involving g has in fact no measurable effect on d/D, and that the conditions of interest fall into this range, we omit the parameter involving g and arrive at the same simpler conclusion as before, but only after due payment in effort for our lack of insight.[2]

1.2.3Advantage in its applications: Dimensional analysis reduces the number of variables and minimizes work:

Dimensional analysis reduces the number of variables that must be specified to describe an event. This often leads to an enormous simplification. In our example of the impacting ball the answer depends on five independent variables (equation 1.2.9), that is, a particular event may be represented as a distribution of d defined in a five-dimensional space of independent variables. Suppose we set out to obtain the answer in a certain region (a certain volume) of this variable-space, by either computation or experimentation, and decide that 10 data points will be required in each variable, with the other four being held constant. This would require obtaining 105 data points. Dimensional analysis, however, shows us that in dimensionless form the answer depends only on two similarity parameters. This two-dimensional space can be explored with similar resolution with only 102 data points, that is, with 0.1% of the effort.[2]

1.3 MATHEMATICAL MODELING

The Dimensional Analysis technique has been used in various applications in most experimentally based areas of physical sciences and engineering. The basic theme of Dimensional Analysis is the Buckingham – л Theorem. The Dimensional Analysis actually follows the fundamental units of relevant quantities. The number of dimensional product is the difference between number of variables and the rank of matrix.[1]

Dimensional analysis is most useful when a mathematical model is not known. Mathematical models of the simple pendulum are well known, and we will use them to generate numerical data and to show how dimensional analysis can be applied to a mathematical model.[8]

CONCLUSION

Dimensional analysis is a simplest and most appropriate tool to make the mathematical modelling of man machine system. By identifying the dependant and independent parameters of the Ambar Charkha, Dimensional Analysis will give us the relationship between them and very easily we can study the effect of the parameters on the system by varying one by one.

REFERENCES

[1] Ahmad S. Ahmad, M. Syazwan Othman, Syed Abdul Kader Ai-Junid, Saad Ahmed,“Mathematical Relationship Development Between Esdd And Rainfall For Contaminated Insulators At Kapar Power Station” Organized By University Tenaga National, Bangi, Selangor, Malaysia; 28-30 August 2006. Pp. 6-11.

[2] Ain A. Sonin, “The Physical Basis of Dimensional Analysis.”

[3] Dehoog, “Why Are Simple Models Often Appropriate In Industrial Mathematics?” F.R. 18th World Imacs / Modsim Congress, Cairns, Australia 13-17 July 2009, Pp. 23-36

[4] Donal P. Finn And Padraig Cunningham, “Physical Model Generation In PDE Analysis Using Model-Based Case-Based Reasoning International Workshop on Qualitative Reasoning About Physical Systems”, Pp90-97, Nara, Japan, June 1994.

[5] G. A. Vignaux,“Dimensional Analysis in Data Modelling”, June 1991, Pp. 1-9.

[6] Prof. G. V. Thakre, Prof. S. G. Patil, D. N. Agrawal, “Assessment of Factors affecting The Productivity ofamber Charkha And Ergonomicevaluation of Workers”, International Journal of Engineering Science And Technology (IJEST), Vol. 3 No.11 November 2011, Pp. 7869-7883

[7] Gunnar Johannsen, Alexander H. Levis, Henk G. Stassen, “Theoretical Problems In Man-Machine Systems Andtheir Experimental Validation” June 1992 Gmu/C3i.T24-P

[8] James F. Pricea,“Dimensional Analysis of Models and Data Sets”, Pp. 437-448.

[9] K.S.Zakiuddin, J.P.Modak, “Formulation of Data Based Ann Model For Thehuman Powered Fodder – Chopper”, Journal of Theoretical and Applied Information Technology, Islamabad Pakistan 31st May 2010. Vol-15, No.2.

[10] Tomas Haggard Jensen,“Assessing Mathematical Modelling Competency”, Vol. 71(5), Pp. 141-148.