# Multipartite Measure of Entanglement

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I introduce an operationally meaningful measure of correlation for multipartite states. This measure can detect entanglement for any multipartite state even if it has no genuine multipartite entanglement and is applicable for multipartite systems of arbitrary dimensions. I will provide an information theoretic interpretation of our measure in terms of quantum mutual information for pure states. It allows us to find the physical reason behind perfect teleportation and superdense coding through W-class states. I will generalize this concepts to form a hierarchy of correlation measures for multipartite states.

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## Introduction

Entanglement is one of the most profound inventions of quantum information theory. For bipartite systems, it plays key role in different information processing tasks [1,2] as an useful resource. The quantification rule [3-7] for entanglement is well defined for bipartite states[8,9] however there are some difficulties in calculating different measures of entanglement for mixed states [9-11]. For multipartite system the problem is more complex as there are many different ways [12-27] of defining multipartite entanglement. It is really difficult to measure quantum correlation responsible for non-local behavior of multipartite systems. e.g., there is no unique way to define a maximally entangled states for multipartite states. Several attempts were made to investigate the entanglement behavior of multipartite systems because of its importance in realizing different aspects like, quantum cryptography, remote information concentration, super-activation, capacities of quantum channels, etc. In this letter I introduce a new measure for multipartite states. This measure enables us to provide the main physical reason behind perfect teleportation and superdense coding through W-class states. It has an information theoretic interpretation in terms of mutual information for pure states. We generalize our concepts to form a hierarchy of correlation measures for multipartite states.

## Correlation Measure for pure state

I introduce first a new measure of correlation for pure multipartite states. Consider a multipartite pure state $|\psi\rangle$ shared between $n$ number of parties. I define the measure of correlation as

$$\mathcal{E} = \max_{\{k\}}\{E(|\psi\rangle_{k:n-k})\},$$

where $1 \leq k \leq \left[\frac{n}{2}\right]$ and $E(|\psi\rangle_{k:n-k})$ denotes the bipartite entanglement of the state $|\psi\rangle$ in the $k{:}\,n-k$ cut. $E$ can be taken as any bipartite entanglement measure. For mixed states, our correlation measure can be extended via convex roof. $\mathcal{E}$ vanishes for fully separable states and it is invariant under local unitary(LU). Also it does not increase by applying local operation and classical communications. These properties simply follows from the properties of its inherent bipartite entanglement measure $E$. Our measure can detect entanglement even for the states with no genuine multipartite entanglement. For example, we can consider three partite bi-separable states. In this case $\mathcal{E}$ vanishes iff states are bi-separable w.r.t all three bi-partitions, i.e., fully separable. Hence this type of correlation measure can be useful in detecting the presence of global entanglement as well as local entanglement (shared between subsystems) in multipartite system. I will explicitly investigate the nature of this correlation for some class of states in three and four qubit systems.

Three qubit pure states can be divided into six classes. For three qubit fully separable class of states, clearly $\mathcal{E} = 0$. In case of three bi-separable classes of states (i.e.$A - BC, B - CA, C - AB$) the maximum value of $\mathcal{E}$ reaches 1 if any two of the three parties share a two-qubit maximally entangled state. For higher dimensional bi-separable system similar conclusion holds. Next, I consider a generic form of three qubit pure state as ,

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

where $\lambda_i \geq 0$, $\sum \lambda_i^2 = 1$ and $\theta \in [0, \pi]$. This class is referred to as 'generic' class, as W-class of states is of measure zero and can be easily obtained from this by substituting $\lambda_4 = 0$. Entanglement of formation for pure bipartite state is the Von-Neumann entropy of reduced density matrix. After a little bit of simplification I can write,

$$\begin{aligned} E(|\psi\rangle_{A:BC}) &= H(\lambda, 1-\lambda), \\ E(|\psi\rangle_{B:AC}) &= H(\mu, 1-\mu), \\ E(|\psi\rangle_{C:AB}) &= H(\nu, 1-\nu), \end{aligned}$$

where
$$\lambda = \frac{1}{2}\left(1 + \sqrt{1 - 4{\lambda_0}^2(1 - {\lambda_0}^2 - {\lambda_1}^2)}\right),$$

$$\mu = \frac{1}{2}\left(1 + \left(1 - 4\left[{\lambda_0}^2({\lambda_3}^2 + {\lambda_4}^2) + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4\cos\theta\right]\right)^{\frac{1}{2}}\right),$$

$$\nu = \frac{1}{2}\left(1 + \left(1 - 4\left[{\lambda_0}^2({\lambda_2}^2 + {\lambda_4}^2) + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4\cos\theta\right]\right)^{\frac{1}{2}}\right)$$

and $H(\cdot)$ denotes the Shannon's binary Entropy. Hence for a three qubit generic pure state,

$$\begin{aligned} \mathcal{E} &= \max\{E(|\psi\rangle_{A:BC}), E(|\psi\rangle_{B:CA}), E(|\psi\rangle_{C:AB})\}, \\ &= \max\{H(\lambda, 1-\lambda), H(\mu, 1-\mu), H(\nu, 1-\nu)\}. \end{aligned}$$

Maximum value of $\mathcal{E}$ over three qubit generic class occurs for the usual GHZ state. In this case $\mathcal{E} = 1$. For three qubit pure symmetric states(ex. generalized GHZ class) the maximization in definition of $\mathcal{E}$ is not necessary as it will show same value in every bi-partition. So the maximum value of $\mathcal{E}$ over three qubit symmetric states, can be obtained by considering any one bipartition. For generalized GHZ state $z_0|000\rangle + z_1|111\rangle$ with $|z_0|^2 + |z_1|^2 = 1$ I have $\mathcal{E} = -H(|z_0|^2, 1 - |z_0|^2)$. Clearly, for this class $\mathcal{E}_{max} = 1$ and this value occurs when $|z_0| = |z_1| = \frac{1}{\sqrt{2}}$. Similarly for generalized W-state $z_0|001\rangle + z_1|010\rangle + z_2|100\rangle$ with $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$ I got $\mathcal{E} = \max\{H(|z_0|^2, 1 - |z_0|^2), H(|z_1|^2, 1 - |z_1|^2), H(|z_2|^2, 1 - |z_2|^2)\}$. Maximum value 1 occurs here when $|z_0| = \frac{1}{\sqrt{2}}$ or, $|z_1| = \frac{1}{\sqrt{2}}$ or, $|z_2| = \frac{1}{\sqrt{2}}$.. This result of generalized W-state indicates the possibility of teleporting unknown qubit and superdense coding as it has maximum value of $\mathcal{E} = 1$. Agrawal *et.al.* showed the perfect teleportation and superdense coding with the W-class state of the above kind. Since to implement perfect qubit teleportation or superdense coding, one need at least one ebit of entanglement, our result shows the main physical reason behind such protocols.

In four qubit system, firstly, I consider the class $X = \{\sum_{j=0}^{3} z_j\, u_j : z_j \in R\}$, where $u_j$ are the tensor product of two same type of 2-qubit Bell state. This class is subset of four qubit generic class. Here I have to consider four 1:3 types of bipartite cuts and three 2:2 types of bipartite cuts. Surprisingly, $E(|\psi\rangle_{1:3}) = 1$ for all four 1:3 partite cuts and for 2:2 cuts we have,

$$E(|\psi\rangle_{AB:CD}) = \sum_{i=0}^{3} |z_i|^2 \log_2 |z_i|^2.$$

$$E(|\psi\rangle_{AC:BD}) = -1/4(|z_0+z_1+z_2+z_3|^2\log_2|z_0+z_1+z_2+z_3|^2/4 + |z_0+z_1-z_2-z_3|^2\log_2|z_0+z_1-z_2-z_3|^2/4 + |z_0-z_1+z_2-z_3|^2\log_2|z_0-z_1+z_2-z_3|^2/4 + |z_0-z_1-z_2+z_3|^2\log_2|z_0-z_1-z_2+z_3|^2/4),$$

$$E(|\psi\rangle_{AD:BC}) = -1/4(|z_0+z_1+z_2-z_3|^2\log_2|z_0+z_1+z_2-z_3|^2/4 + |z_0+z_1-z_2+z_3|^2\log_2|z_0+z_1-z_2+z_3|^2/4 \; |z_0-z_1+z_2+z_3|^2\log_2|z_0-z_1+z_2+z_3|^2/4 + |z_0-z_1-z_2-z_3|^2\log_2|z_0-z_1-z_2-z_3|^2/4).$$

Hence one is a lower bound of $\mathcal{E}$ and the maximum value of this correlation for the above class of states is 2. Maximum value of $\mathcal{E}$ for this class is attained when $z_i = \frac{1}{2} \quad \forall i$ or $z_i = 1$ for any one $i \in \{0,1,2,3\}$. I have found that maximum value of $\mathcal{E}$ for this class of states occurs for 'Cluster States' (ex. $\frac{1}{2}[|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle]$ and its LU equivalent states). Whereas maximum value of $\mathcal{E}$ for both 4-qubit generalized GHZ and W class states is 1.

## Another New Correlation Measure

Now, I define another new measure of correlation in terms of mutual information as

$$\mathcal{I} = \max_k\{I(|\psi\rangle_{k:n-k})\},$$

where $1 \le k \le \left[\frac{n}{2}\right]$ and $|\psi\rangle_{k:n-k}$ denotes all possible $k:n-k$ partite cuts and $I(|\psi\rangle_{k:n-k})$ denotes the quantum mutual information. Thus $I$ is a detector of correlation (classical or quantum) in any state. Clearly this definition has a close relation with our previous quanity $\mathcal{E}$.

**Theorem**: *Two measures of correlations* $\mathcal{E} = \max_k\{E(|\psi\rangle_{k:n-k}): 1 \le k \le \left[\frac{n}{2}\right]\}$ *and* $\mathcal{I} = \max_k\{I(|\psi\rangle_{k:n-k}): 1 \le k \le \left[\frac{n}{2}\right]\}$ *satisfies the relation* $\mathcal{E} = \frac{1}{2}\mathcal{I}$ *holds. Here,* $E(\cdot)$, *is any entanglement measure of bipartite states,* $I(\cdot)$ *is quantum mutual information and* $|\psi\rangle$ *is any n-partite pure state.*

Proof: If I take Entanglement of formation $(E_f)$ as a measure of bipartite entanglement then,

$$\mathcal{E} = \max_k\{E(|\psi\rangle_{k:n-k}) : 1 \le k \le \left[\frac{n}{2}\right]\},$$

$$= \max_k\{E_f(|\psi\rangle_{k:n-k}) : 1 \le k \le \left[\frac{n}{2}\right]\}$$

Now, let us denote $\rho_X = tr_1(|\psi\rangle_{k:n-k}\langle\psi|)$, $\rho_Y = tr_2(|\psi\rangle_{k:n-k}\langle\psi|)$ and $\rho = |\psi\rangle_{k:n-k}\langle\psi|$ where $tr_i(.)$ denotes trace over $i$ th part, $i = 1,2$(i.e. trace over first $k$ or last $n-k$ parties of the n-partite state $|\psi\rangle_n\langle\psi|$). Also denote $S(.)$ as Von-Neumann entropy. Then,

$$\begin{aligned}
\mathcal{I} &= \max_k\{I(|\psi\rangle_{k:n-k}) : 1 \le k \le \left[\frac{n}{2}\right]\} \\
&= \max_k\{S(\rho_X) + S(\rho_Y) - S(\rho) : 1 \le k \le \left[\frac{n}{2}\right]\} \\
&= \max_k\{S(\rho_X) + S(\rho_Y) : 1 \le k \le \left[\frac{n}{2}\right]\} \\
&= \max_k\{2S(\rho_X) : 1 \le k \le \left[\frac{n}{2}\right]\} \\
&= \max_k\{2E_f(\rho) : 1 \le k \le \left[\frac{n}{2}\right]\} \\
&= 2\mathcal{E}
\end{aligned}$$

Here, I have used the fact that von-Neumaan entropy of a pure state is zero and for a bipartite pure state any good measure of entanglement reduces to Von-Neumann entropy of the reduced density matrices. Here I have taken entanglement of formation but any other measure would do. Hence the maximum value of $\mathcal{I}$ for different qubit system behaves like Von-Neumann entropy. $\mathcal{I}$ attains maximum (global) value 2 for GHZ state $(\frac{1}{2}(|000\rangle + |111\rangle))$ and generalized W state$(z_0|001\rangle + z_1|010\rangle + z_2|100\rangle$; for $|z_0|^2 + |z_1|^2 = \frac{1}{2}$ and $|z_2|^2 = \frac{1}{2})$ in three qubit system. For four qubit generalized GHZ and W class of states, maximum value for $\mathcal{I}$ reaches 2 and $\mathcal{I}$ attains the maximum value 4 (global maximum for four qubit system) for cluster states.

The above construction could be further generalized to find other measures of correlation. We first denote $\mathcal{E}$ in equation ([1]) by $\mathcal{E}_2$. Let us consider a $n$-partite pure state. We now define

$$\mathcal{E}_3(|\psi\rangle) = \max_{\text{tri-partite entanglement}}\{\mathcal{E}_2(|\psi\rangle)\}$$

i.e. we introduce two cuts among the parties who share the state and treat it as tripartite state. The maximum is taken over the correlation values $\mathcal{E}_3$ of all such tripartite states. For an $n$-partite state we can introduce upto $n-1$ cuts. So we can recursively define

$$\mathcal{E}_{n-1}(|\psi\rangle) = \max_{(n-1)\text{-partite entanglement}}\{\mathcal{E}_{n-2}|\psi\rangle\}$$

Therefore we can have a hierarchy of correlation measures for a $n$-partite state upto $\mathcal{E}_{n-1}$. *Examples*: We consider 4-qubit GHZ-state $|\psi_4^{GHZ}\rangle$. This state is symmetric under exchange of any two or more parties. Therefore it is enough to consider one state each from 1: 3 and 2: 2 cut. It turns out that for all such bipartite cuts $\mathcal{E}_2 = 1$. There can be 6 types of tripartite cuts. But again due to symmetry, it is enough to consider only one type of tripartite cut i.e. A:B:CD. and it is easily found that $\mathcal{E}_3 = 1$. Now we consider 4-qubit W-state $|\psi_4^W\rangle$. This state also shows similar symmetry like GHZ-state. In this case $\mathcal{E}_2 = \max_{1\le k\le 4} H\left(\frac{k}{4}, \frac{4-k}{4}\right)$. Maximum value is attained for $k = 2$ and maximum value is 1. Among all tripartite cuts maximum value of the correlation is 1 i.e. $\mathcal{E}_3 = 1$. However for 4-qubit case we get $\mathcal{E}_2 = 1, \mathcal{E}_3 = 2$ for cluster states.
Further, we consider an important multipartite state from qubit system.

$$|S(n,k)\rangle = \frac{1}{\sqrt{\binom{n}{k}}} \sum_{\text{permutations}} |\underbrace{0\ldots0}_{n-k}\underbrace{1\ldots1}_{k}\rangle$$

This is Dicke state of $n$ particles with $k$ excitations. Clearly $|S(n,1)\rangle = |\psi_n^W\rangle$. Also $|S(n,n-k)\rangle = \sigma_x^{\otimes n}|S(n,k)\rangle$ i.e. they are local unitarily equivalent hence they both have similar entanglement hierarchy. For $n = 4$ we consider

$$|S(4,2)\rangle = \frac{1}{\sqrt{6}}[|0011\rangle + |1001\rangle + |0101\rangle + |1100\rangle + |1010\rangle + |0110\rangle]$$

This is also a symmetric state. Introducing bipartite cut the form of the state reduces to the form

$$\frac{1}{\sqrt{6}}|\alpha\rangle|a\rangle + \sqrt{\frac{2}{3}}|\beta\rangle|b\rangle + \frac{1}{\sqrt{6}}|\gamma\rangle|c\rangle$$

where $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$ is a orthonormal set and $\{|a\rangle, |b\rangle, |c\rangle\}$ is another orthonormal set. Hence $\mathcal{E}_2(|S(4,2)\rangle) \approx 1.25$ and it is higher than both GHZ- and W-state. Also for this state, $\mathcal{E}_3(|S(4,2)\rangle) \approx 1.25$. fully separable state of $n$-qubit has $\mathcal{E}_2 = \mathcal{E}_3 = \ldots = \mathcal{E}_{n-1} = 0$. Hence according to the hierarchy in 4-qubit

$$\text{Fully Separable} < \text{GHZ or W} \leq \text{Dicke state} \leq \text{Cluster state.}$$

For mixed states we use the convex roof extension of the corresponding quantities i.e. we define

$$\mathcal{E}_2(\rho) \quad = \min_{\{p_i,|\psi_i\rangle\}}\{\sum_i p_i\, \mathcal{E}_2(|\psi_i\rangle)\}$$

An evident lower bound of this quantity can be defined as

$$\min_{\{p_i,|\psi_i\rangle\}}\{\max_{k:n-k,1\leq k\leq\left[\frac{n}{2}\right]}\sum_i p_i\, \mathcal{E}_f(|\psi_i\rangle_{k:n-k})\}$$

The minimum is over all pure state decomposition of $\rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|$. This construction for mixed state can be further extended to $\mathcal{E}_{(n-1)}(\rho)$ for a $n$-partite mixed state in similar way like pure state.

The construction of other correlation measure is similar to the above. We denote $\mathcal{I}$ in equation ([8]) by $\mathcal{I}_2$ and then for a $n$-partite state we can define $\mathcal{I}_3, \cdots, \mathcal{I}_{n-1}$ similarly.

## Conclusion

In conclusion, we have introduced a new measure of correlation for multipartite systems which is able to detect entanglement of a multipartite state in arbitrary dimensions. We have explicitly checked its behavior for pure three qubit and four qubit states. We find an information theoretic interpretation of our newly introduced measure. This measure could be used to explain the teleportation and superdense coding through W-class states. Analyzing the behavior of our correlation measures, we hope this work will help us to formulate a general paradigm of understanding multipartite systems in future and will provide us to find new information theoretic tasks.

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