A STUDY OF UNIFIED MULTIVARIABLE INTEGRALS AND LAPLACE TRANSFORMS ASSOCIATED WITH THEM

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ABSTRACT: Multivariable integral evaluated here involve the exponential function, generalized polynomials and multivariable H-function This multivariable integral is unified, useful and most general in nature and capable of yielding a large number of integral and multivariable Laplace transforms as their special cases.

INTRODUCTION

The multivariable H- function occurring in the paper will be defined and represented in the following form [5,pp,251-252,eqn.(C.1)-(C.3)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H_{p,q\,:\,p_1,q_1;\,\dots\dots;\,p_r,q_r}^{o,n\,:\,m_1,n_1;\,\dots\dots;\,m_r,n_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;\ \alpha_j^{(1)},\dots,\alpha_j^{(r)})_{1,p};\ (c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\dots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j;\ \beta_j^{(1)},\dots,\beta_j^{(r)})_{1,q};\ (d_j^{(1)},\delta_j^{(1)})_{1,q_1};\dots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L1}\dots\int_{Lr}\psi(\varepsilon_1,\dots,\varepsilon_r)\,\emptyset_1(\varepsilon_1)\dots\emptyset_r(\varepsilon_r)z_1^{\varepsilon_1}\dots z_r^{\varepsilon_r}d\varepsilon_1\dots d\varepsilon_r$$

(1.1)

For the convergence conditions of the integral given by (1.1) and other details of the multivariable H- function we refer to the book by Srivastava et al . [5,pp,252-253,eqns.(C.4)-(C.8)]

For r = 2, the above multivariable H-function reduces to the well known H-function of two variables (Srivastava et al. 1982, p. 82)

The generdized polynomials $S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_s \end{bmatrix}$ occurring here in will be defined and represented in the following form which differs slightly from that given by Srivastava [4, p. 185, eqn. (7)].

$$S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_s \end{bmatrix} = S_{N_1,\dots,N_s}^{M_1,\dots,M_s}[x_1,\ \dots x_s]$$

$= \sum_{k_1=0}^{[N_1/M_1]} \ldots \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1 k_1}}{k_1!} \ldots \frac{(-N_s)_{M_s k_s}}{k_s!} A[N_1, k_1; \ldots ; N_s, k_s] x_1^{k_1} \ldots x_s^{k_s}$

(1.2)

Where $N_i = 0, 1, 2, \ldots ; M_i \neq 0 \; [i = 1, \ldots , s]$.

M_i is an arbitrary positive integer and the coefficients $A[N_1, k_1; \ldots ; N_s, k_s]$ are arbitrary constants, real or complex.

If we take s = 1 in the equation (1.2) and denote A[N, k] thus obtained by $A_{N,k,}$ we arrive at the well know general class of polynomials $S_N^M\ [x]$ introduced by Srivastava [3, p. 1, eqn. (1)].

The double laplace transform occurring herein will be defined and represented in the following manner.

$L\{f(x_1, x_2); s_1, s_2\} = \int_0^\infty \int_0^\infty e^{-s_1 x_1 - s_2 x_2} f(x_1, x_2) dx_1 dx_2$

(1.3)

Result 1:

$$\int_0^\infty \int_0^\infty (\lambda'_1 x_1 + \lambda'_2 x_2)^{\sigma_1 - 1} (\lambda''_1 x_1 + \lambda''_2 x_2)^{\sigma_2 - 1} \exp\left[-s_1(\lambda'_1 x_1 + \lambda'_2 x_2) - s_2(\lambda''_1 x_1 + \lambda''_2 x_2)\right]$$

$$S_{N_1}^{M_1}[C(\lambda'_1 x_1 + \lambda'_2 x_2)^{\rho_1}]\ S_{N_2}^{M_2}[D(\lambda''_1 x_1 + \lambda''_2 x_2)^{\rho_2}]\ H_{p,q,:\,p_1,q_1,p_2,q_2}^{o,n:m_1,n_1,m_2 n_2}\left[\begin{matrix} z_1(\lambda'_1 x_1 + \lambda'_2 x_2)^{v_1} \\ z_2(\lambda''_1 x_1 + \lambda''_2 x_2)^{v_2} \end{matrix}\right|$$

$$\left.\begin{matrix} \left(a_j, \alpha_j^{(1)}, \alpha_j^{(2)}\right)_{1,p} : \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,p_1} ; \left(c_j^{(2)}, \gamma_j^{(2)}\right)_{1,p_2} \\ \left(b_j, \beta_j^{(1)}, \beta_j^{(2)}\right)_{1,q} : \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,q_1} ; \left(d_j^{(2)}, \delta_j^{(2)}\right)_{1,q_2} \end{matrix}\right] dx_1 dx_2$$

$$= \frac{1}{k} \sum_{k_1=0}^{[N_1/M_1]} \sum_{k_2=0}^{[N_2/M_2]} \frac{(-N_1)_{M_1 k_1} (-N_2)_{M_2 k_2}}{k_1!\ k_2!\ s_1^{(\sigma_1 + \rho_1 k_1)} s_2^{(\sigma_2 + \rho_2 k_2)}}$$

$$H_{p,q,:\,p_1+1,q,\ ;p_2+1,q_2}^{o,n\,:\,m_1,n_1+1;\,m_2 n_2+1}\left[\begin{matrix} z_1 s_1^{-v_1} \\ z_2 s_2^{-v_2} \end{matrix}\right| \begin{matrix} \left(a_j, \alpha_j^{(1)}, \alpha_j^{(2)}\right)_{1,p} : \\ \left(b_j, \beta_j^{(1)}, \beta_j^{(2)}\right)_{1,q} : \end{matrix}$$

$$\left.\begin{matrix} (1 - \sigma_1 - \rho_1 k_1, v_1) \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,p_1} ; (1 - \sigma_2 - \rho_2 k_2, v_2) \left(c_j^{(2)}, \gamma_j^{(2)}\right)_{1,p_2} \\ \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,q_1} ; \qquad \left(d_j^{(2)}, \delta_j^{(2)}\right)_{1,q_2} \end{matrix}\right] dx_1 dx_2$$

(2.1)

$$\text{Where } k = \begin{vmatrix} \lambda'_1 & \lambda''_1 \\ \lambda'_2 & \lambda''_2 \end{vmatrix} \neq 0, v_i > 0, \text{Re}(s_i) >= 0, \left[\text{Re}(\sigma_i) + v_i \min_{1 \leq j \leq m_i} \text{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0, \text{ i=1,2}$$

Prof of (2.1) : To Prove (2.1) we first express the H- function of two variables occurring in the left hand side of (2.1) in term of Mellin-Barnes type of contour integrals the interchange the order of ξ_1, ξ_2 and x_1, x_2 integrals we get in following result after little simplification.

$$\frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_s} \psi(\xi_1, \xi_2) \phi_1(\xi_1) \phi_2(\xi_2)\, z_1^{\xi_1} z_2^{\xi_2} d\xi_1 d\xi_2\, \boldsymbol{\Delta} \quad (2.2)$$

Where in (2.2)

$$\boldsymbol{\Delta} = \int_0^\infty \int_0^\infty (\lambda'_1 x_1 + \lambda'_2 x_2)^{v_1\xi_1 + \sigma_1 - 1} (\lambda''_1 x_1 + \lambda''_2 x_2)^{v_2\xi_{2+}\sigma_2 - 1} \exp\left[-s_1(\lambda'_1 x_1 + \lambda'_2 x_2) - s_2(\lambda''_1 x_1 + \lambda''_2 x_2)\right]$$

$$S_{N_1}^{M_1}[C(\lambda'_1 x_1 + \lambda'_2 x_2)^{\rho_1}] S_{N_2}^{M_2}[D(\lambda''_1 x_1 + \lambda''_2 x_2)^{\rho_2}] dx_1 dx_2 \quad (2.3)$$

Now we evaluate Δ in following manner:

We have [(7)Wider 1989, p. 241, eqn. (7)]

$$\int_0^\infty \int_0^\infty F(\lambda'_1 x_1 + \lambda'_2 x_2, \lambda''_1 x_1 + \lambda''_2 x_2) dx_1 dx_2 = \frac{1}{k} \int_0^\infty \int_0^\infty F(u_1, u_2)\, du_1 du_2 \quad (2.4)$$

Where k stands for the expression mentioned in (2.1)

If we take $F(\lambda'_1 x_1 + \lambda'_2 x_2, \lambda''_1 x_1 + \lambda''_2 x_2) = f_1(\lambda'_1 x_1 + \lambda'_2) f_2(\lambda''_1 x_1 + \lambda''_2 x_2)$

Then (2.4) Transformed to

$$\int_0^\infty \int_0^\infty f_1(\lambda'_1 x_1 + \lambda'_2 x_2,) f_2(\lambda''_1 x_1 + \lambda''_2 x_2) dx_1 dx_2 = \frac{1}{k} \int_0^\infty f_1(u_1) du_1 \int_0^\infty f_2(u_2) du_2 \quad (2.5)$$

Consider $f_1(\lambda'_1 x_1 + \lambda'_2 x_2) = (\lambda'_1 x_1 + \lambda'_2 x_2)^{v_1\xi_1 + \sigma_1 - 1} \exp[-s_1(\lambda'_1 x_1 + \lambda'_2 x_2)]\, S_{N_1}^{M_1}[C(\lambda'_1 x_1 + \lambda'_2 x_2)^{\rho_1}]$

$f_2(\lambda''_1 x_1 + \lambda''_2 x_2) = (\lambda''_1 x_1 + \lambda''_2 x_2)^{v_2\xi_{2+}\sigma_2 - 1} \exp[-s_2(\lambda'_1 x_1 + \lambda'_2 x_2)]\, S_{N_2}^{M_2}[D(\lambda'_1 x_1 + \lambda'_2 x_2)^{\rho_2}]$

Then form (2.5) we get

$$\int_0^\infty \int_0^\infty (\lambda'_1 x_1 + \lambda'_2 x_2))^{v_1\xi_1 + \sigma_1 - 1} (\lambda''_1 x_1 + \lambda''_2 x_2)^{v_2\xi_{2+}\sigma_2 - 1} \exp[-s_1(\lambda'_1 x_1 + \lambda'_2 x_2) - s_2(\lambda'_1 x_1 + \lambda'_2 x_2)]$$

$$S_{N_1}^{M_1}[C(\lambda'_1 x_1 + \lambda'_2 x_2)^{\rho_1}] S_{N_2}^{M_2}[D(\lambda''_1 x_1 + \lambda''_2 x_2)^{\rho_2}] dx_1 dx_2$$

$$=\frac{1}{k}\int_0^\infty u_1^{v_1\xi_1+\sigma_1-1}\, e^{-s_1u_1} S_{N_1}^{M_1}\,[Cu_1^{\rho_1}]\, du_1 \int_0^\infty u_2^{v_2\xi_2+\sigma_2-1}\, e^{-s_2u_2} S_{N_2}^{M_2}\,[Du_2^{\rho_2}]\, du_2$$
(2.6)

On expressing the general class of polynomials occurring on the right hand side of (2.6) in terms of series with the help of (1.2) interchanging the order of integrals and summation in the result thus obtained and interchanging the u_1 and u_2 integrals .with the help of know formula Gradshteyn and Ryzhik[(2), p. 317, eqn. (3.38), (14)] eqn. (2.2) takes the Following form :

$$\frac{1}{k}\sum_{k_1=0}^{[N_1/M_1]}\sum_{k_2=0}^{[N_2/M_2]}\frac{(-N_1)_{M_{1k_1}}(-N_2)_{M_{2k_2}}}{k_1!\ k_2!}$$

$$\{\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi_1)\phi_2(\xi_2)\psi(\xi_1,\xi_2)\frac{\Gamma(\sigma_1+\rho_1k_1+v_1\xi_1)\times\Gamma(\sigma_2+\rho_2k_2+v_2\xi_2)}{s_1^{\sigma_1+\rho_1k_1+v_1\xi_1}s_2^{\sigma_2+\rho_2k_2+v_2\xi_2}}z_1^{\xi_1}z_2^{\xi_2}d\xi_1 d\xi_2\}$$

(2.7)

Finally on reinterpreting the Mellin-Barnes integral occurring in right hand side of (2.7) in termsH- function of two variables we get the desired results (2.1).

Main Result:

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{r}\left(\lambda_1^{(i)}x_1+\cdots+\lambda_r^{(i)}x_r\right)^{\sigma_i-1}\right)$$

$$\exp\,[-\textstyle\sum_{i=1}^{r} s_i\left(\lambda_1^{(i)}x_1+\cdots+\lambda_r^{(i)}x_r\right)]$$

$$S_{N_1,\ldots,N_r}^{M_1,\ldots,M_r}\begin{bmatrix} e_1\left(\lambda_1^{(1)}x_1+\cdots+\lambda_r^{(1)}x_r\right)^{v_1} \\ \vdots \\ e_r\left(\lambda_1^{(r)}x_1+\cdots+\lambda_r^{(r)}x_r\right)^{v_r}\end{bmatrix}$$

$$H\begin{bmatrix} z_1\left(\lambda_1^{(1)}x_1+\cdots+\lambda_r^{(1)}x_r\right)^{\rho_1} \\ \vdots \\ z_r\left(\lambda_1^{(r)}x_1+\cdots+\lambda_r^{(r)}x_r\right)^{\rho_r}\end{bmatrix} dx_1\cdots dx_r$$

$$=\frac{1}{k'}\sum_{k_1=0}^{|N_1/M_1|}\cdots\sum_{k_r=0}^{|N_r/M_r]}\left(\prod_{i=1}^{r}\frac{e_i(-N_i)_{M_ik_i}}{k_i!}\ s_i^{-(\sigma_i+v_i\,k_i}\right)$$

$$A[N_1,k_1;\ldots;N_r,k_r]\, H^{o,n:\; m_1,n_1+1;\ldots,\, m_r,r+1}_{p,q:\; p_1+1,q_1;\ldots,\, p_r+1,q_r}\left[\begin{matrix} z_1 & s_1^{-\rho_1} \\ & \vdots \\ z_1 & s_r^{-\rho_r}\end{matrix}\right|$$

$$\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p'}: (1-\sigma_1+v_1 k_1,\rho_1),\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;$$

$$\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q'}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;$$

$$\left.\begin{matrix}(1-\sigma_r+v_r k_r,\rho_r),\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_r} \\ \left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \qquad (3.1)$$

$$\text{Where } K'\begin{vmatrix}\lambda_1^{(1)}\lambda_2^{(1)}\ldots\lambda_r^{(1)} \\ \lambda_1^{(2)}\lambda_2^{(2)}\ldots\lambda_r^{(2)} \\ \vdots \quad \vdots \quad \vdots \\ \lambda_1^{(r)}\lambda_2^{(r)}\ldots\lambda_r^{(r)}\end{vmatrix}\neq 0$$

$$(v_i,\rho_i)\geq 0 (i=1,\ldots,r) \text{ (not all zero simultaneously)}$$

$$Re(s_i)>0, Re\,[\sigma_i+\rho_i+\min_{1\leq j\leq m_i}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)]>0,(i=1,\ldots,r)$$

Proof of (3.1)

On making use of the result given below (which is a r-variable analogue of (2.2))

$$\int_0^\infty\ldots\int_0^\infty F\left(\lambda_1^{(1)}x_1+\cdots+\lambda_r^{(1)}x_r,\ldots\ldots,\lambda_1^{(r)}x_1+\cdots+\lambda_r^{(r)}x_r\right)dx_1\cdots dx_r$$

$$=\frac{1}{k'}\int_0^\infty\ldots\int_0^\infty F(u_1\ldots\ldots,u_r)du_1\cdots du_r \qquad (3.2)$$

Taking the definition of generalized polynomial given by (1.2) into consideration and proceeding in a manner indicated earlier in the proof of (2.1), we arrive at the desired result (3.1) after a little simplification.

SPECIAL CASES:

If we put $\lambda_1^{(1)}=\lambda_2^{(2)}=\cdots=\lambda_r^{(r)}=1$ in (3.1) and each of remaining λ involved therein equal to zero, we get the following interesting multidimensional Laplace transformation involving the product of the generalized polynomials and the multivariable H-function.

$$L\{(\prod_{i=1}^r x_i^{\sigma_i-1})S^{M_1,\ldots,M_r}_{N_1,\ldots,N_r}\left[\begin{matrix}e_1 & x_1^{v_1} \\ & \vdots \\ e_r & x_r^{v_r}\end{matrix}\right] H\left[\begin{matrix}z_1 & x_1^{\rho_1} \\ & \vdots \\ z_r & x_r^{\rho_r}\end{matrix}\right]; s_1,\cdots s_r\}$$

$$\sum_{k_1=0}^{[N_1/M_1]}\cdot\cdot\cdot\sum_{k_r=0}^{[N_r/M_r]}\left(\prod_{i=1}^r\frac{e_i(-N_i)_{M_i k_i}}{k_i!}\; s_i^{-(\sigma_i+v_i k_i)}\right) \qquad (3.3)$$

$$A[N_1,k_1;\;...\;;N_r,k_r]\;H^{****}$$

where H**** function occurring in (3.3) stands for the same multivariable H- function which occurs on the right hand side of (3.1)

If we put in (2.1) $M_1 = M_2 = s_1 = s_2 = 1, \rho_1 = \rho_2 = C = D = 1$ and $v_1 = v_2 = 0$ and replace A_{N_1,k_1} by $\binom{N_1+\alpha_1}{N_1}$ and A_{N_2,k_2} by $\binom{N_2+\alpha_2}{N_2}$ respectively , then $S_{N_1}^{M_1}, S_{N_2}^{M_2}$ occurring therein reduce to Laguerre polynomials Srivastava and Singh [(6), p.159, eqn (1.8)] and reduces the two variable H-function to unity Srivastava at al. [(5), p. 82] we arrive at a known result Dhawan [1, p. 417, eqn (2.2)] after a little simplification.

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