A STUDY OF UNIFIED FINITE INTEGRAL INVOLVING THE MULTIVARIABLE SPECIAL FUNCTIONS

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ABSTRACT: In this paper we establish finite integral which involve the product of the extended Jacobi polynomial, the Generalized Polynomials and the Multivariable H-function on account of the general nature of the function and polynomials occurring in these integral, my findings provide interesting unifications of a large number of (new and known) results.

Keywords: Beta function, H- function, Hermite Polynomial**,** Hyper Geometric function, Jacobi polynomials

INTRODUCTION:

To unify the classical orthogonal polynomials Viz.Jacobi,Hermite and LaguerreFujiwara[2] defined a class of generalized classical polynomials by means of following Rodrigues formula:

$$Rn(X)=\frac{(-1)^n k^n}{n!(x-p)^{\beta}(q-x)^{\alpha}}\frac{d^n}{dx^n}[(x-p)^{\beta+n}(q-x)^{\alpha+n}], p<x<q, \alpha>-1, \beta>-1 \quad (1.1)$$

Denote these polynomials by Fn $(\beta, \alpha; x)$ and call them extended Jacobi polynomials Thakare [9] obtained the following form of Rn(x)=Fn$(\beta, \alpha; x)$

$$Fn(\beta,\alpha;x)=\frac{(-1)^n K^n (q-x)^n (1+\beta)_n}{n!} 2F1\left[\begin{matrix}-n, -n-\alpha; & \frac{p-x}{q-x}\\ 1+\beta; & \end{matrix}\right] p<x<q \quad (1.2)$$

Fujiwara [2] proved that when p=-1,q=1 and k=$\frac{1}{2}$

$$Fn(\beta,\alpha;x) = P_n^{(\alpha,\beta)}(x)$$

Where $P_n^{(\alpha,\beta)}(x)=\frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n 2F1\left[\begin{matrix}-n, -n-\alpha; & \frac{x+1}{x-1}\\ 1+\beta; & \end{matrix}\right]$ is Jacobi polynomial [4] (1.3)

The multivariable H- function occurring in the paper will be defined and represented in the following form [6,pp,251-252,eqn.(C.1)-(C.3)]

$$H\begin{bmatrix}z_1\\ \vdots\\ z_r\end{bmatrix}=H^{o,n:m_1,n_1;\ldots\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j^{(1)},\ldots\ldots,\alpha_j^{(r)})_{1,p};\\ (b_j;\beta_j^{(1)},\ldots\ldots,\beta_j^{(r)})_{1,q};\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L1}\ldots\int_{Lr}\psi(\varepsilon_1,\ldots,\varepsilon_r)\,\emptyset_1(\varepsilon_1)\ldots\emptyset_r(\varepsilon_r)z_1^{\varepsilon_1}\ldots z_r^{\varepsilon_r}d\varepsilon_1\ldots d\varepsilon_r \quad (1.4)$$

For the convergence conditions of the integral given by (1.4)and other details of the multivariable H- function we refer to the book by Srivastava et al . [6,pp,252-253,eqns.(C.4)-(C.8)]

The generdizedpolynomials$S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\begin{bmatrix}x_1\\ \cdot\\ \cdot\\ \cdot\\ x_s\end{bmatrix}$occurring hereinwillbedefinedandrepresented inthe following form which differs slightly from that given by Srivastava [7, p. 185, eqn. (7)].

$$S_{N_1,\ .\ .\ .\ .\ ,\ N_s}^{M_1,\ .\ .\ .\ .\ ,\ M_s}\begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_s \end{bmatrix} = S_{N_1,\ .\ .\ .\ .\ ,\ N_s}^{M_1,\ .\ .\ .\ .\ ,\ M_s}[x_1,\ \ .\ .\ .x_s]$$

$$= \sum_{k_1=0}^{[N_1/M_1]} . . . \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1k_1}}{k_1!} . . . \frac{(-N_s)_{M_sk_s}}{k_s!} A[N_1, k_1;\ .\ .\ .\ ;\ N_s, k_s] x_1^{k_1} . . . x_s^{k_s} \quad (1.5)$$

Where $N_i = 0, 1, 2,\ .\ .\ .; M_i \neq 0\ [i = 1,\ .\ .\ .\ , s]$.

$M_i$is an arbitrary positive integer and the coefficients $A[N_1, k_1;\ .\ .\ .\ ;\ N_s, k_s]$ are arbitrary constants, real or complex.

If we take s = 1 in the equation (1.5) and denote A[N, k] thus obtained by $A_{N,k,}$ we arrive at the well know general class ofpolynomials$S_N^M\ [x]$introduced by Srivastava [8, p. 1, eqn. (1)].

PRELIMINARIES:

In this paper we need the following results :

(i) [[1], p.10, eq.(13) Viz]$\int_b^a (t-b)^{x-1}(a-t)^{y-1}\,dt = (a-b)^{x+y-1}\,B(x,y), Re(x) > 0, Re(y) > 0, b < a$ (2.1)

Where B(x,y) is beta function .

(ii) The Hyper Geometric function [4]

2F1(a, b; c; z)

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n\ n!} \quad (2.2)$$

MAIN INTEGRAL

$$\int_p^q (x-p)^t\,(q-x)^\alpha Fn(\beta,\alpha;\ x) S_{N_1,\ .\ .\ .\ .\ ,\ N_s}^{M_1,\ .\ .\ .\ .\ ,\ M_s}\begin{bmatrix} e_1(x-p)^{u'_1}(q-x)^{v'_1} \\ \vdots \\ e_s(x-p)^{u'_s}(q-x)^{v'_s} \end{bmatrix} H \begin{bmatrix} z_1(x-p)^{u_1}(q-x)^{v_1} \\ \vdots \\ z_r(x-p)^{u_r}(q-x)^{v_r} \end{bmatrix} dx$$

$$= \frac{(-1)^n K^n (1+\beta) n (q-p)^{t+\alpha+n+1}}{n!} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (-n-\alpha)_\ell}{\ell!\,(1+\beta)_\ell}$$

$$\sum_{k_1=0}^{\left[\frac{N_1}{M_1}\right]} . . . \sum_{k_s=0}^{\left[\frac{N_s}{M_s}\right]} A[N_1, k_1;\ .\ .\ .\ ;\ N_s, k_s] \prod_{j=1}^{s} \left(\frac{(-N_j)_{M_{jk_j}} e_j^{k_j}}{k_j!} (q-p)^{(u'_j+v'_j)}\right)$$

$$H\ \begin{matrix} 0, n+2 & : & * \\ p+2, q+1 & : & * \end{matrix} \left[\begin{matrix} z_1(q-p)^{u_1+v_1} \\ . \\ . \\ . \\ . \\ z_r(q-p)^{u_r+v_r} \end{matrix}\ \middle|\ \begin{matrix} A & : & * \\ B & : & * \end{matrix}\right]$$

Where

A=(-t-ℓ-$u'_1 k_1$ ….. $-u'_s k'_s; u_l, \dots, u_r$)(-α-n+ℓ-$v'_1 k_l$ ….. $-v'_s k_s; v_1, \dots, v_r$)($a_j; \alpha_j^{(1)}, .\ .\ .\ \ , \alpha_j^{(r)}$)1,p

$$B= \left(b_j;\ \beta_j^{(1)} \ldots\ldots \beta_j^{(r)}\right)_{1,q}(-\alpha-n-t-1-(u'_1+v'_1)k_l-\ldots-(u'_s+v'_{s,})k_s; u_1+v_1, \ \ldots, u_r+v_r) \qquad (3.1)$$

Also the asterisk (*) occurring in the right hand side of (3.1) indicates that parameters at these places are the same as the multivariable H-function in (1.4).

The integral (3.1) Is valid under the following conditions :

(i) $(u_i, v_i u'_j, v'_j) \geq 0 \ \ i = 1, \ldots, r$ and j=1, . . s(not all zero simultaneously)

(ii) $\mathrm{Re}(t+\ell)+\sum_{i=1}^{r} u_i \min_{1\leq j\leq m_i}\left[Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$

(iii) $\mathrm{Re}(\alpha+n-\ell)+\sum_{i=1}^{r} v_i \min_{1\leq j\leq m_i}\left[Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$

Proof of (3.1)

To establish (3.1) replace the multivariable H- function by its Mellin-Barnes contour integral from. Now we interchange the order of x and $\varepsilon_1 \ldots\ldots \varepsilon_r$ integrals (which is permissible under the conditions stated with (3.1) in the result thus obtained and get after a little simplification the left hand side of (3.1)) (sayΔ) as:

$$\Delta= \frac{1}{(2\pi\omega)^r}\int_{L1} \ldots\ldots\ldots\ldots \int_{Lr} \emptyset_1(\varepsilon_1)\ldots\ldots\emptyset(\varepsilon_r)\psi(\varepsilon_1\ldots\ldots\varepsilon_r){z_1}^{\varepsilon_1}\ldots\ldots\ldots {z_r}^{\varepsilon_r}$$

$$\{\int_p^q (x-p)^{t+u_1\varepsilon_1+u_r\varepsilon_r}(q-x)^{\alpha+v_1\varepsilon_1+\cdots+v_r\varepsilon_r}$$

$$\mathrm{Fn}(\beta,\alpha;x) S_{N_1,\ \ldots,\ N_s}^{M_1,\ \ldots,\ M_s}\begin{bmatrix} e_1(x-p)^{u_1}(q-x)^{v_1} \\ \vdots \\ e_s(x-p)^{u's}(q-x)^{v's}\end{bmatrix} dx\}\ d\varepsilon_1 \ldots\ldots d\varepsilon_r \quad (3.4)$$

Now in the inner integral (3.4) put the value of extended Jacobi polynomial Fn(β,α;x) from (1.2) and generalizedpolynomial [1.5], interchange the order of integration and summation (which is permissible under the condition stated with (3.1)). The equation (3.4)takes the following form after a little simplification with the help of known result (2.1)

$$\Delta= \frac{(-1)^n K^n (1+\beta)_n (q-p)^{t+\alpha+n+1}}{n!}\sum_{l=0}^{\infty}\frac{(-1)^l(-n-\alpha)_l}{l!\,(1+\beta)_l}$$

$$\sum_{k_1=0}^{[N_1/M_1]} \ldots \sum_{k_s=0}^{[N_s/M_s]} A[N_1, k_1;\ \ldots;\ N_s, k_s]\prod_{j=1}^{s}\left(\frac{(-N_j)_{M_j k_j} e_j^{k_j}}{k_j!}(q-p)^{(u'_j+v'_j)}\right)$$

$$\left\{\frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\ldots\int_{L_r} \emptyset_1(\varepsilon_1)\ldots\emptyset_r(\varepsilon_r)\psi(\varepsilon_1\ldots.\varepsilon_r) z_1^{\varepsilon_1}\ldots. z_r^{\varepsilon_r}(q-p)^{\sum_{i=1}^{r}(u_i+v_i)\varepsilon_i}\frac{\left[(t+l+\sum_{j=1}^{s}u'_j k'_j+\sum_{i=1}^{r}u_i\varepsilon_i+1)\right[(\alpha+n-l+\sum_{j=1}^{s}u'_j k'_j+\sum_{i=1}^{r}v_i\varepsilon_i+1)}{\left[(2+\alpha+n+t+\sum_{i=1}^{r}(u_i+v_i)\varepsilon_i+\sum_{j=1}^{s}(u'_j+v'_j)k_j\right.}\right\}$$

$$d\varepsilon_1 \ldots d\varepsilon_r (3.5)$$

Finally on reinterpreting the multiple Mellin-Barnes contour integral occurring in right hand side of (3.5)in terms of the multivariable H-function We arrived at the desired result (3.1).

Special case :If we take p=-1,q=1 and $k=\frac{1}{2}$In (3.1) we get the result which is same as obtained by Gupta and Pawan [3,p.84,eqn.(2.7)].

Further if we take s=1 in (3.1) and further put $N_1 = 0$in the general class of polynomial thus obtained we arrive at the result SaxenaandRamawat [5,p.158,eqn.(2.4)].

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