Assessing The Influence Of Vector Representational Formats On Students Understanding Of Vector Concepts

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ABSTRACT

Teachers use vector representations suitably with pronounced flexibility, both to communicate information about physical phenomena and as effective tools for problem solving. Student representational competence is vital to achieve success in conceptual understanding and in solving problems. Nonetheless, much depends on how learners have been introduced to vectors in mathematics and physics classes and how they internalise vector operations. We have tested the student understanding of basic vector concepts both in math and physics contexts and our study reveals significant learning gaps and that the understanding of vector operations and concepts is influenced by the semiotic representations of vectors. We present the indexing of the vector representations in the TUV, the questions framed in math context by the authors and the results of our study on selected concepts.

INTRODUCTION

Precise understanding of the properties and operations related to semiotic representation/s of a mathematical object is one of the key aspects of developing skills related to representational transformation [1]. Mieke De Cock [2] states two categories of skills relevant to the use of external representations namely representational fluency and representational flexibility. The author quotes that representational fluency involves the ability to interpret or construct representations, as well as the ability to translate and switch between representations (on demand) accurately and quickly while representational flexibility involves making appropriate representational choices in a given learning or problem solving situation. In-depth investigations on how students decipher information in specific representation/s have revealed that students possess different competence levels while processing vector representations. In the context of vector addition, Hawkins et.al [3] have investigated through interviews, students' consistency of graphical vector addition method on two-dimensional vector addition tasks and observed that the students adopted one method and persisted with its use despite changes in the visual representations. Hawkins.et.al. [4] have analysed students' responses to the aligned and divergent representations of a vector addition question; with and without grid and infer that small scale changes in the arrangement of vectors affect the solution methods. Barniol & Zavala [5] have investigated the effect of context and position of vectors on addition procedures adopted by students in two sets of problems: three isomorphic and three different representations (three different arrangements of vectors on a grid). The authors have enumerated the vector representations by students and catalogued the types of errors in problems stated in both the sets. Heckler and Scaife [6] have found that students in an introductory physics course had problems with vector addition and subtraction (in one dimension (1D) and in two dimensions (2D)) in the arrow format in contrast to the *ijk* format. Properties and operations related to arrows are difficult for students. Use of technology in modern day science teaching has been explored to supplement learning methods related to the arrow format of a vector representation. Meera & Shubha [7] designed simulation activities to bring to focus the functional understanding of vector arrow representation of electric field. DurgaPrasad et.al [8] have used an interactive computational system named TFV (Touchy Feely Vectors), which connects the algebra and the geometric representations involved in vectors, and allows students to learn addition and resolution of vectors in an interactive way.

RATIONALE FOR THE STUDY

Several studies have found that college students who have been trained on vector concepts have shortcomings in understanding both vector properties and operations. Results show similar error features on a few vector concepts and differences on a few others. It is significant to gain insights into students' understanding of vector concepts in an Indian context as they are trained in a different learning environment. Regardless of the stage of learning, it is crucial to test the vector knowledge of students, including basic concepts. In addition, a vector has many semiotic representations. Therefore, understanding vector concepts is intertwined with the features of the representational formats. Literature survey suggests that there is limited research work reported in this domain that investigates students' vector representational fluency. In our study, we have chosen the Test of Understanding Vectors (TUV) developed by Barniol and Zavala which is established to be a reliable assessment tool on vector concepts [9]. TUV assesses more vector concepts than other previous tests on vectors. Furthermore, the questions comprise of additional vector representations than were tested in previous studies. Hence, we found the assessment tool to be a good choice for our research study which was directed at testing students on both the components of representational fluency (students' ability to interpret representations and to translate between representations effectively). Earlier studies investigated the multiple-choice test of understanding of vectors based just on concepts (TUV) by applying item response theory (IRT)[10] and the Rasch model [11].

METHODOLOGY

This study was carried out with three cohorts of physics graduate students from three consecutive batches enrolled in the graduate (Masters') course. A total of 217 students (male and female) volunteered to be participants of test. There were no pre-determined criteria for participation in the test. However, students had completed undergrad (Bachelors') course from different undergraduate institutions which ensured heterogeneity of the sample. We used the test module furnished as a supplement material in English by Barniol and Zavala [9]. TUV is a comprehensive 20-item test in Multiple Choice Question (MCQ) format with five options for each test item, used for the assessment of students' understanding of basic concepts in vector algebra. Students were facilitated to complete the test individually at their own pace and were allowed to use calculators, if needed.

In this study, we investigate the student understanding of vector concepts by a comprehensive analysis of the responses to the test items of TUV. We scrutinise and index the 20 test items of TUV based on the vector representation in the question and option format as a matrix shown in table 1.The vector representations in all test items are indicated in the horizontal row while the vector representations in options are indicated in the vertical column. The numbers in the grid indicate the TUV item number. For instance, in item 1 of TUV, the vector representation in question is arrow-on-grid and vector representation in the option is arrow-on-grid.

Table 1. Index of the 20 multiple choice test items of TUV based on vector representation in question and option format.

Question format ➡ Option format ⬇	*Arrow*	*Arrow-on-grid*	*Graphical(with x and y coordinate axes)*	*ijk*
Arrow-on-grid		1, 5, 13, 19	4 , 9	
Graphical (with x and y coordinate axes)			2, 11	10
ijk				8 , 15
Word-statement	3, 12	7, 16		
equation-based	6, 18		14	
Numerical				17, 20

We analyse the test responses based on the vector representation in both the question and options presented for the question (besides vector representation in question and a numerical answer) and provide a detailed analysis of the affordance/s of the representation. Further, we formulate additional questions on select vector concepts which have an altered vector representation in the question/option format. A few questions are free-ended while the others are multiple-choice questions. We analyse the responses to the supplementary questions and also investigate the influence of the representation in understanding vector properties and operations. The additional questions formulated by us were administered to 67 students from the original group 217 students. Similar to the larger group, there were no pre-specified criteria for selection of the subsample.

Analysis of TUV results and responses to questions framed by us on the basis of vector representations

In this section, we analyse the student responses to TUV aligned on the vector representations mentioned above. The numbers in tables II – VI indicate the percentage of responses evaluated to the test item for each option. The number in bold indicates the percentage of correct responses.

Arrow-on-grid & Arrow-on-grid

Table II shows the percentage of students who selected a particular choice for the test items which had the arrow-on-grid representation of a vector, both in question and in options.

Table II.

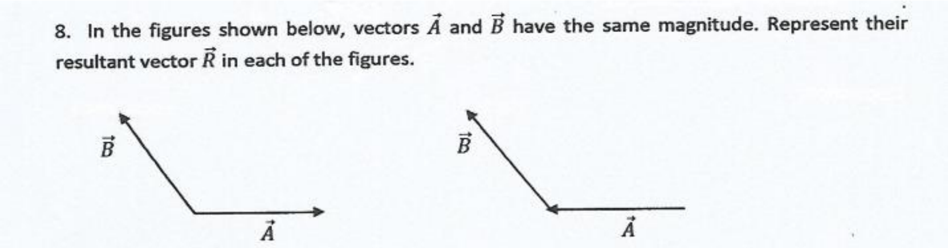
Item	Vector concept	A	B	C	D	E
5	Direction of a vector	18	7	**59**	6	10
1	Vector addition in 2D	12	31	3	43	**11**
13	Vector subtraction in 2D	21	18	9	41	**11**
19	Vector subtraction in 1D	6	52	20	11	**11**

In item 5, the basis to choose vectors which have the same direction essentially is vectors being 'parallel'. For those who have chosen option A, ascertaining the parallel nature among the options may be elusive as vectors $\vec{K}$ & $\vec{L}$ appear seemingly parallel. In fact, options $\vec{F}$ & $\vec{G}$ to problem 2 of Vector Concept Quiz constructed by Nguyen and Meltzer [12] have a greater semblance of being 'parallel vectors' and hence confusion about how to recognize when two vectors are parallel did result in the dominant wrong choice by students. Among the two options $\vec{F}$ & $\vec{G}$, one option has been reconfigured in TUV which is vector $\vec{L}$. Students who chose option E may have confusion between 'equal vectors' and 'same direction vectors'.

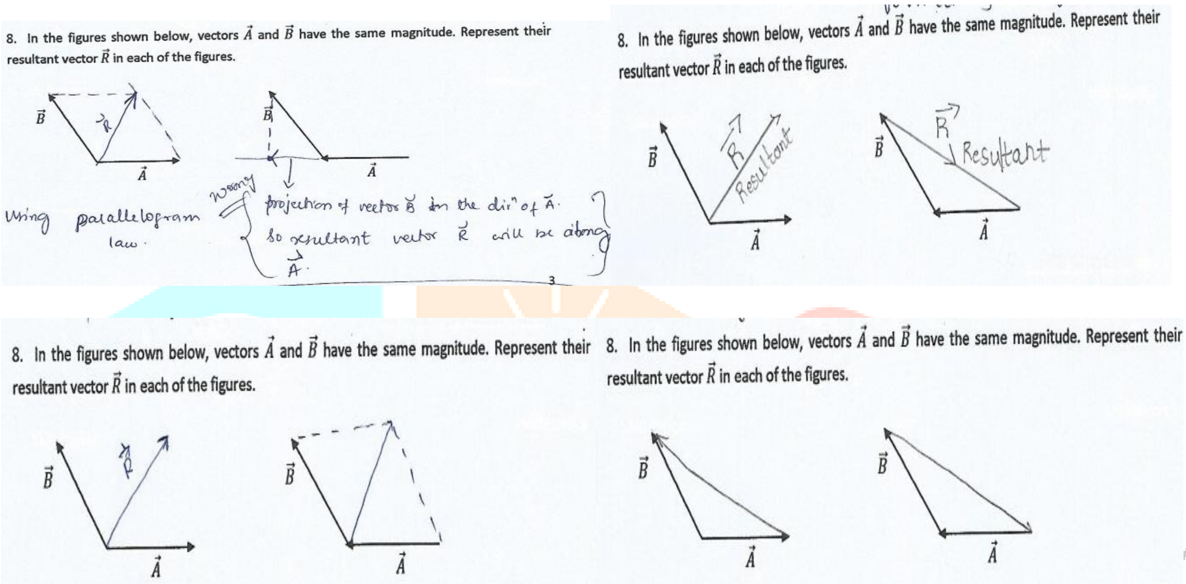
Vector addition and subtraction are basic operations which are to be exercised within a framework of specific algorithms. For item 1, the respondents are required to choose the vector sum from the options presented in the arrow-on-grid format. The solution may be approached either by performing a geometrical construction or by writing the horizontal and vertical components of each vector, subsequently adding them and returning to the arrow-on-grid representation. However, the representation in this test item appears to primarily cue a geometrical construction to find the sum of two vectors. Our students are taught to use parallelogram law (applicable when vectors are aligned tail to tail as two adjacent sides of a parallelogram) and triangle law (applicable when vectors are aligned tip to tail as adjacent sides of a triangle) to find the resultant. The construction approach in our study has mostly been the tip-tip construction obtained by imprecise application of triangle law.

Vector subtraction operation in 2D can be reduced to the addition operation by essentially reversing the direction of the vector to be subtracted. Item 13 relates to difference of two vectors represented as arrows on a grid. The response to this item required an additional layer to vector addition task which is to know the appropriate representation of the negative of vector $\vec{\mathrm{B}}$ in the arrow-on-grid format. Vector difference can be obtained either by using the parallelogram law or triangle law of addition subsequent to drawing the negative of vector $\vec{\mathrm{B}}$. An alternate and a simpler method to choose the correct option were to algebraically subtract the magnitudes of the horizontal and vertical components of the vectors which can be deduced from the grid. Students who chose option D approached the solution by "plug-and-chug" application of the mathematical algorithm to the vector magnitudes and ignored the vector directions, virtually treating this 2-D vector subtraction as in a 1-D example. A similar approach was displayed by majority of the students in 1D subtraction. A higher magnitude of vector $\vec{B}$ appears to influence their idea of 'first' vector in the difference.

Research outcomes in PER have revealed student problems with the arrow representation. Heckler and Scaife [6] investigated student understanding of vector addition and subtraction in both the arrow representation (two variants) and *ijk* notation. The authors found the average performance in the *ijk* format to be better than the performance in the arrow format in either math or physics context. Flores et.al [13] investigated student familiarity with vector addition by administering a question in the arrow format (ref fig1). They observed that some students treated vectors as scalars or followed incorrect addition algorithms. Nyugen & Melzer [12] pose an open-ended 2D vector addition problem, the vectors being non-co-initial (fig 5). While the average percentage of correct responses of students in the calculus based course was about 65%, percentages of correct responses of students in the algebra-based course were in the range of 22% - 44%. Hawkins et.al [3] conducted interviews on 2D vector addition questions based on four different arrow-vector representations. For discussion in their paper, the authors choose two representations: one with two vectors in the tip-to-tip arrangement on a grid (question 8 in [3]) and the other arrangement without a grid (question 5 in [3]). They found that most students stuck to one method for the entire interview despite changes in details that may favour the choice of another method suited to the representation. In another research study, Hawkins et al.[4] probed whether visual representation of vectors can affect the methods students use to add them and found that students tend to use an identical method of vector operation irrespective of the vector position; either on grid or no grid. In our study, the reasons for the weak performance on test items related to vector addition (2D) and subtraction (1D & 2D may be either due to the incorrect application of addition laws (which the students tend to compartmentalise as parallelogram and triangle laws) or due to the unfamiliar arrow-on-grid representation. To validate this conjuncture, we gave the following problem to a subgroup of 67 students from amongst the group of 217 students. We also present some illustrative samples from their responses.



Some illustrative responses of students are provided below.



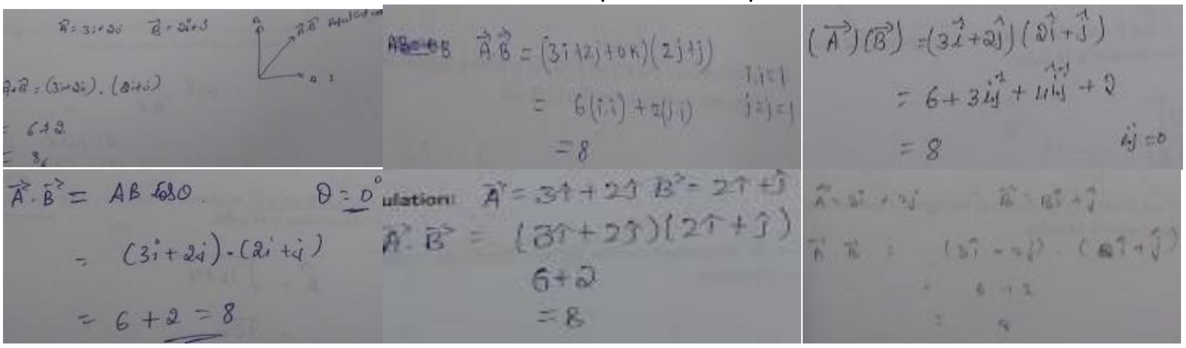
Fifty seven per cent of the students represented the resultant vector correctly by using the parallelogram law of addition (in the first figure of our question) and 38% represented the resultant correctly by using the triangle law of addition (in the second figure). The confusion in the application of addition laws is evident, either in the presence of the grid or absence of the grid.

The aforementioned results lead us to the important aspect of affordances of arrow-on-grid representation. To decipher the parallel nature of vectors in an arrow representation (without grid) would definitely be confusing. The grid, therefore, is expected to facilitate the learner to decipher the components of a vector and hence to calculate the angle of the vector with the horizontal axis. Though our students are unfamiliar with the arrow-on-grid representation, they learn about the graphical representation of a vector i.e. arrow on a coordinate system. However, the absence of coordinate axes may have been a deterrent to write the vectors in the *ijk* format (or write the algebraic sum of horizontal and vertical components individually) and perform addition. Further, the task requires translating the vector sum in either *ijk* format or horizontal and vertical components of the sum to the grid. Vector subtraction too, is difficult in the absence of an initial understanding of negative of a vector. Students are unable to abstract the relevant information such as angle of a vector measured from the horizontal axis and components of a vector, encoded by such a representation. A desirable outcome is that students are able to interpret each vector representation aptly and perform addition of vectors on arrow-on-grid using either component or construction method and perform 'pure'-arrow addition with the application of addition laws (construction method).

We intended to investigate if vector representations in *ijk* format cue addition and subtraction with much ease as they mimic algebraic operations. We therefore provided students with the following problems.

Consider vectors $\vec{A}=3\hat{\imath}+2\hat{\jmath}$ and $\vec{B}=2\hat{\imath}+\hat{\jmath}$. Find the resultant vector.

Fifty two per cent of the students calculated the vector sum correctly, though not really high for the relatively simple task. Majority among students who did not provide the right answer calculated dot product of the two vectors. Probably, the percentage of correct answers would be higher if the students were asked to 'find sum' or asked to 'find $\vec{A}+\vec{B}$'. A few written responses are presented.



Following is the problem presented in the *ijk* format that involves subtraction of vectors including multiplication of a vector by a negative scalar.

Consider vectors $\vec{A} = 2\hat{\imath} + 2\hat{\jmath}$ and $\vec{B} = -\hat{\imath} + 3\hat{\jmath}$. Calculate the vector $\vec{A} - 3\vec{B}$.

Seventy nine per cent of the students calculated the required vector $\vec{A} - 3\vec{B}$ correctly. As expected, the *ijk* format prompts algebraic-like calculations with ease. The use of the difference sign (which is unambiguous) appears to have a positive influence on the calculation of the required vector. However, the most common error by students is to ignore the negative sign for the *y* component of vector $-3\vec{B}$ while evaluating the required vector difference. Better performance on vector addition and subtraction in *ijk* format does not suggest that students have understood the directionality aspect and rules related to vector addition and subtraction.

Graphical & arrow-on-grid

An interesting question and option format is presented in items 4 & 9. Table III shows the vector concepts and percentage of responses evaluated to test items which had the vector in the question as a graphical representation and as arrow-on-grid in options.

Table III

Item	Vector concept	A	B	C	D	E
4	y –component of a vector	19	11	**51**	10	9
9	x –component of a vector	25	14	11	**47**	3

The concept of component of vector is being tested in the two test items with the vector represented on a graph and the components as arrows on a grid. Notation used for writing the components of a vector is either $\vec{A} = \overrightarrow{A_x} + \overrightarrow{A_y}$ with $\overrightarrow{A_x}$ & $\overrightarrow{A_y}$ as the vector components or $\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath}$ with A_x & A_y as the scalar components. The second notation is more often in practice by teachers in the Indian context of classroom teaching. The graphical representation in the two test items aids identification of components A_x & A_y by mere visual inspection. Students, after identifying the scalar components, are required to choose the vector components which are $A_x\,\hat{\imath}$ and $A_y\,\hat{\jmath}$. About fifty per cent of the respondents provided correct responses to both the test items. The dominant incorrect responses to both test items are the components which are greater than the correct component along the axis obtained by rotating the vector $\vec{A}$ towards the corresponding axes. A similar reasoning for this specific choice on the vector component question has been discussed by Barniol and Zavala (ref. question 1 in [14]). Finding the vector components of a vector in arrow representation is not considered synonymous with rotation of the vector! The labelling of the angle Φ with the *y* axis, in principle, is redundant as the options do not comprise of trigonometric functions of the angle in question. However, it proved to be a functional distractor for 11% of the students who chose option C for item 9. The same inference may not be valid for the 11% incorrect responses to item 4 with the choice of option B as the angle marked does not instantly cue the students to look in the direction of the *x* axis. In our study, students who chose the incorrect options are unable to decode the graphical information of the vector into the *x* and *y* components though the origin of the Cartesian coordinates system could be identified. Also, students are required to read representational details from the graph and set aside irrelevant information to prevent its influence on the choice of incorrect answers.

Graphical – Graphical

The table below (table IV) shows the vector concepts and percentage of responses calculated to test items which had the vector in the graphical representation both, in the question and options.

Table IV.

Item	Vector concept	A	B	C	D	E
2	Unit vector	14	41	**12**	6	27
11	Multiplication by a negative number	6	31	**26**	18	19

Item 2 required identification of the graphical representation of unit vector in the direction of an arbitrary vector. For 41% of students, the graphical representation as option B had a categorical influence on their thinking that 'unit' vector in the direction of an arbitrary vector $\vec{A}$ has *x* and *y* components, one unit each. This clearly suggests how a representation can interfere with their thinking and influence a student to opt for an intuitively obvious answer which is eventually incorrect.

To test the procedural knowledge of calculation of unit vector in the direction of a specified vector, we gave the following multiple-choice question.

Consider the vector $\vec{A} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$. *Choose the option that is the unit vector in the direction of vector* $\vec{A}$ *.*

a) $\hat{\imath}+\hat{\jmath}+\hat{k}$
b) *1 unit*
c) $1(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$
d) $\frac{1}{3}(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

We evaluated the percentage of student responses to each option (table V).

Table V

Option	a	b	c	d
Percentage of responses	11	0	6	83

It is evident that understanding of the concept of unit vector in one representation does not necessarily reveal proper understanding in another representation. The graphical representation of unit vector was challenging for students in comparison to the unit vector procedural character associated with *ijk* format.

Item 11 relates to the multiplication of a vector by a negative scalar. This item tests the basic requirement to perform subtraction which is to understand the idea of a negative of a vector. Majority of the students were unable to relate the negative sign associated with vector $3\vec{A}$ to the arrow representation on a graph. The direction of a negative of vector as an arrow needs an intuitive - spatial understanding which is observed to be difficult from a student's point of view. However, negative of a vector multiplied by a scalar in *ijk* format does not compel this understanding. A fairly good percentage of correct answers to our question on vector subtraction (ref: Calculate the vector difference $\vec{A}$-3$\vec{B}$) validates the above statement.

***ijk* –graphical**

Table VI shows the vector concept and percentage of responses calculated for the test item which had the vector in the *ijk* format in question and graphical format as options.

Table VI.

Item	Vector concept	A	B	C	D	E
10	Vector representation	32	**44**	3	2	19

Item 10 required students to map the vector $\vec{A}$ in the *ijk* format to a graph. Less than fifty per cent of the students showed a fluency to switch vector representations effectively. However, to know if the component of the vector $\vec{A}$ in the direction of negative *x* axis was an explicit impediment to students' visualisation of a vector on a graph; we presented the students a vector in the *ijk* format with both its components in the directions of positive *x* axis and positive *y* axis

Summary and conclusion

We re-examined the responses to test items of TUV with focus on the semiotic representations of a vector in both the question and answer-options. The analysis revealed an inadequate grasp of vector concepts which were tested through the specific vector representations in TUV. In the context of arrow-on-grid representation, students failed to recognise the affordance/s of the representation (i.e. writing the components or marking of relevant axes) which would assist in the task of vector addition. Another affordance of arrow-on-grid representation is that of calculation of the angle of the vector with a horizontal axis to identify parallel vectors. Understanding of the spatial orientation of negative of a vector is crucial to perform vector subtraction in either arrow format or arrow-on-grid format. We observe that the graphical representation (with a defined origin and indicated coordinate axes) does not foster the spatial understanding of negative of a vector to the desired level, though students pursuing science courses use graphical representations extensively in math and physics learning. This inadequate understanding hinders operations related to vector arrow subtraction. However, operations with vectors in *ijk* format appear to be easy from our analysis of responses to test items of TUV and to the additional questions formulated by us, as they mimic algebraic procedures. The question is whether this notable performance helps us conclude that *ijk* representation is well understood. Students do experience difficulty in representing a vector in the *ijk* format as an arrow on a graph. Decoding a graphical representation of a vector into *ijk* format is also arduous. Students are required to make spatial reasoning of vectors in the *ijk* format irrespective of the sign of the components to enhance their ability to switch between representations. The arrow representation which is introduced as the first format is presumed to be understood effortlessly. This introduction seems to be inexplicit as our students experienced difficulties with the arrow representation which was evident from the responses to the question formulated by us (ref question 8). The variations in spatial arrangement of two vectors represented as arrows seem to confuse the students while applying addition rules. As a particular arrangement, the orthogonal representation of vectors as arrows offers the affordance of application of Pythagorean Theorem without any hassle as the students are trained in the use of appropriate procedures in many instances. Moreover, the choice of incorrect responses in evaluation of dot product and magnitude of a vector indicate a flawed deduction that a vector operation necessarily yields a result which is a vector.

Previous research has revealed a key aspect that influences the success rate and the solution strategy is the representational format in which the question is asked [15] and significant performance differences between different representations [2, 16, 17]. In our study, we found the performance on the TUV test items and the formulated questions in altered vector representations alarmingly inadequate though these concepts are used by students at various levels of their learning. Hence research of the type documented in this article is essential. In the introductory classes of teaching vectors, each representation and its associated mathematical methods are introduced in a cursory manner. In addition, there is very little emphasis on the correlation between the heterogeneous (semiotic) representations of a vector which lays the foundation for attaining representational fluency (involves the ability to translate between representations). A holistic understanding of vector principles and operations is through achieving a threshold of representational fluency. Attaining this fluency is two layered: firstly, to understand the strengths and limitations of each semiotic representation and secondly to understand the distinctive association between representations. Students should be taught to act on each representation effectively. Details in a representation such as direction of a vector in arrow format, negative sign in *ijk* format and angle in graphical representation, need attention in deciphering information it conveys. This is important as vector operations are the processes by which information in a representation is elicited and manipulated. Vector operations related to each basic concept need to be internalised in different semiotic representations to strengthen representational fluency. While we agree that there has to be a robust understanding of vector concepts, we also emphasise the interplay of understanding concepts with the different representations of a vector, as one isn't independent of the other. It is not a question if the concept being tested is elementary. What is relevant is how students process novel representations. A weak understanding of the essential components of representations is bound to affect the robustness of their conceptual framework and needs to be addressed at the introductory level.

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