The point of coincidence and common fixed point for a pair of mappings with $(\psi, \alpha, \beta)-$ weak contraction in cone metric space

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Abstract

In this paper, we establish results involving common fixed points and point of coincidence for two self-mappings satisfying (ψ, α, β) −weak contractive condition in cone metric space.

Introduction

Banach contraction principle is a very important tool for solving existence problems in many branches of mathematics. This contraction principle has further several generalizations in metric spaces as well as cone metric spaces. Huang and Zhang [10] introduced the concept of cone metric space, where every pair of elements is assigned to an element of a Banach space and defined a partial order on the Banach space with the help of a subset of the Banach space called cone which satisfy certain properties. In the same work Huangand Zhang established some fixed point theorems for cone metric spaces. Further many authors have given several generalizations of that theorems. Some of which were established with the help of weak contraction in cone metric spaces.

Weak contraction was introduced by Alber et al. [14] for Hilbert spaces and subsequently extended to metric spaces by Rhodes [1]. In particular Choudhury et. al. [3, 4] established some fixed point resultsin cone metric spaces with the help of two control functions ψ and φ. In the present work, we established three fixed point results for two self- mappingin cone metric spaces with the help of three different control functions ψ, α and β.

Before coming to our main result we give some preliminaries of cone metric space which was firstly introduced by Huang and Zhang [10].

1 Mathematical preliminaries

Definition 1.1 *[10]Let E be a real Banach space and θ is the zero of the Banach space E. Let P be a subset of E. P is called a cone if*

(i) P *is closed, non-empty and* $P \neq \{\theta\}$

(ii) $ax + by \in P$ *for all* $x, y \in P$ *and non-negative real numbers* a, b

(iii) $P \cap (-P) = \{\theta\}$

For a given cone P we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. Here $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $x - y \in intP$; where $intP$

denotes the interior of P. $x \leq y$ is same as $y \geq x$ and $x \ll y$ is same as $y \gg x$.

A cone P is called normal if there is a real number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. The cone is called regular if every increasing and boundedabove sequence $\{x_n\}$ in E is convergent. Equivalently the cone P is regular if and only if every decreasing and bounded below sequence is convergent.

Definition 1.2 *[10] Let X be a non-empty set. Suppose the mapping*

$$d : X \times X \rightarrow E \text{ satisfies:}$$

(i) $\theta \leq d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = \theta$ *if and only if* $x = y$

(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

Then d *is called a cone metric on* X *and* (X,d) *is called a cone metric space.*

Definition 1.3 *[10] Let* (X,d) *be a cone metric space,* $\{x_n\}$ *a sequence in* X *and* $x \in X$. *For every* $c \in E$ *with* $<< c$ *; we say that* $\{x_n\}$ *is:*

(i) a Cauchy sequence if there is a natural number N *such that for all* $n, m > N$; $d(x_n, x_m) << c$

(ii) convergent to x *if there is a natural number* N *such that for all* $n > N$; $d(x_n, x) << c$ *for some* $x \in X$.

Definition 1.4 *[10] If* P *is a normal cone, then* $\{x_n\}$ *converges to* x *if and only if* $d(x_n, x) \longrightarrow \theta$ *as* $n \rightarrow \infty$. *Further, in this case* $\{x_n\}$ *is a Cauchy sequence if and only if* $d(x_n, x_m) \rightarrow \theta$ *as* $n, m \rightarrow \infty$.

Definition 1.5 *[10]* (X,d) *is called a complete cone metric space if everyCauchy sequence in* X *is convergent.*

Lemma 1.1: [5] If P is a normal cone in E then

(i) If $\theta \leq x \leq y$ and $\theta \leq a$, where a is a real number, then $\theta \leq ax \leq ay$.

(ii) If $\theta \leq x_n \leq y_n$, for $n \in N$ and $\lim_n y_n = y$, then $\theta \leq x \leq y$.

Lemma 1.2: [8] If E be a real Banach space with cone P in E, then for $a, b, c \in E$

(i) If $a \leq b$ and $b << c$, then $a << c$.

(ii) If $a << b$ and $b << c$, then $a << c$.

Lemma 1.3: [9] Let E be a real Banach space with cone P in E, then P is normal if and only if $x_n \leq y_n \leq z_n$ and $\lim_n x_n = \lim_n z_n = x$ imply $\lim_n y_n = x$.

Lemma 1.4: [4] Let (X,d) be a complete cone metric space with regular cone P such that $d(x,y) \in intP$, for $x, y \in X$ with $x \neq y$. Let $\beta : intP \cup \{\theta\} \rightarrow intP \cup \{\theta\}$ be a function with the following properties:

(i) $\beta(t) = \theta$ if and only if $t = \theta$

(ii) $\beta(t) << t$, for $t \in intP$ and

(iii) either $\beta(t) \leq d(x,y)$ or $d(x,y) \leq \beta(t)$, for $t \in intP \cup \theta\}$ and $x,y \in X$

Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing.Then $\{d(x_n, x_{n+1})\}$ is convergent to either $r = \theta$ or $r \in intP$.

Definition 1.6 *[7] Let T and S be self-maps of a set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S, and w is called a point of coincidence of T and S. Self-maps T and S are said to be weakly compatible if they commute at their coincidence point; that is, if $Tx = Sx$ for some $x \in X$, then $TSx = STx$.*

Lemma 1.5: [11] Let T and S be weakly compatible self-maps of a set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.

2 Results

Theorem 2.1 *Let (X,d) be a cone metric space with regular cone P such that $d(x,y) \in intP$, for $x,y \in X$ with $x \neq y$. Let $T,S : X \to X$ are mappings satisfying the inequality*

$$\psi(d(Tx,Ty)) \leq \alpha(d(Sx,Sy)) - \beta(d(Sx,Sy)) \quad \text{for } x,y \in X \tag{1}$$

Where $\psi, \alpha, \beta : intP \cup \{\theta\} \to intP \cup \{\theta\}$ are such that ψ is continuous andmonotone non decreasing, α and β are continuous functions with

(i) $\psi(t) = \alpha(t) = \beta(t) = \theta$ *if and only if* $t = \theta$;

(ii) $\psi(t) - \alpha(t) + \beta(t) > \theta$ *for all* $t > \theta$;

(iii) $\beta(t) << t$ *for* $t \in intP$;

(iv) *either* $\beta(t) \leq d(x,y)$ *or* $d(x,y) \leq \beta(t)$ *for* $t \in intP \cup \{\theta\}$ *and* $x,y \in X$.

If $T(X) \subseteq S(x)$ and $S(X)$ is a complete subspace of X, then T and S havea unique point of coincidence in X. Moreover if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof: Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, we can construct a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1}$, for all $n \geq 0$. If there exists an integer $N \geq 0$ such that $Tx_N = Tx_{N+1}$, then $Sx_{N+1} = Tx_{N+1}$, which means that T and Shave a point of coincidence and we have nothing to prove. So we will assume that $Tx_N \neq Tx_{N+1}$, for all $n \geq 0$.

Now substituting $x = x_{n+1}$ and $y = x_n$ in equation (1), for all $n \geq 0$ we have,

$$\psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \leq \alpha\big(d(Sx_{n+1}, Sx_{n+2})\big) - \beta\big(d(Sx_{n+1}, Sx_{n+2})\big)$$

i.e;

$$\psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \leq \alpha\big(d(Tx_n, Tx_{n+1})\big) - \beta\big(d(Tx_n, Tx_{n+1})\big) \tag{2}$$

Now, for all $n \geq 1$, we have,

$$\begin{aligned}
&\psi\big(d(Tx_n, Tx_{n+1})\big) - \psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \\
&\geq \psi\big(d(Tx_n, Tx_{n+1})\big) - \alpha\big(d(Sx_{n+1}, Sx_{n+2})\big) + \beta\big(d(Sx_{n+1}, Sx_{n+2})\big) \\
&\geq \psi\big(d(Tx_n, Tx_{n+1})\big) - \alpha\big(d(Tx_n, Tx_{n+1})\big) + \beta\big(d(Tx_n, Tx_{n+1})\big) \\
&\geq \theta
\end{aligned}$$

This implies $\psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \leq \psi\big(d(Tx_n, Tx_{n+1})\big)$. That is $d(Tx_{n+1}, Tx_{n+2}) \leq d(Tx_n, Tx_{n+1})$, for all $n \geq 0$.

This implies that the sequence $\{d(Tx_n, Tx_{n+1})\}$ is monotone decreasing and bounded below by θ, so it must converge because the cone P is regular.

So there exists $r \in intP \cup \{\theta\}$ such that

$$d(Tx_n, Tx_{n+1}) \to r \text{ as } n \to \infty.$$

Now taking $n \to \infty$ in equation (2) and using the continuity of ψ, α and β, we have

$$\psi(r) \leq \alpha(r) - \beta(r) \Rightarrow \psi(r) - \alpha(r) + \beta(r) \leq \theta \Rightarrow r = \theta,$$

$$i.e; \qquad \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = \theta \qquad (3)$$

Now we show that $\{Tx_n\}$ is a Cauchy sequence. If not then, there exists some $c \in E$ with $\theta << c$ such that for all $n_0 \in N$, there exist $n, m \in N$ with $n > m > n_0$ such that

$$d(Tx_m, Tx_n) << c \Rightarrow d(Tx_m, Tx_n) \not<< \beta(c).$$

Hence by property (iv) of β in the theorem, we have $\beta(c) \leq d(Tx_m, Tx_n)$.

Therefore, there exist sequences $m(k)$ and $n(k)$ in N such that for all positive integers k, $n(k) > m(k) > k$ and $d\left(Tx_{m(k)}, Tx_{n(k)}\right) \geq \beta(c)$.

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d\left(Tx_{m(k)}, Tx_{n(k)}\right) \geq \beta(c) \qquad (4)$$

$$\text{and} \qquad d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) << \beta(c) \qquad (5)$$

Now, $\beta(c) \leq d\left(Tx_{m(k)}, Tx_{n(k)}\right) \leq d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) + d\left(Tx_{n(k)-1}, Tx_{m(k)}\right)$

$$\Rightarrow \beta(c) \leq d\left(Tx_{m(k)}, Tx_{n(k)}\right) \leq \beta(c) + d\left(Tx_{n(k)-1}, Tx_{m(k)}\right)$$

Taking $n \to \infty$ in the above inequality and using Lemma (1.3)

$$\lim_{k\to\infty} d\left(Tx_{m(k)}, Tx_{n(k)}\right) = \beta(c) \qquad (6)$$

Also by triangular inequality, for all $k \geq 0$, we have

$$d\left(Tx_{m(k)+1}, Tx_{n(k)+1}\right) \leq d\left(Tx_{m(k)+1}, Tx_{m(k)}\right) + d\left(Tx_{m(k)}, Tx_{n(k)}\right) + d\left(Tx_{n(k)}, Tx_{n(k)+1}\right) \qquad (7)$$

and

$$d\left(Tx_{m(k)}, Tx_{n(k)}\right) \leq d\left(Tx_{m(k)}, Tx_{m(k)+1}\right) + d\left(Tx_{m(k)+1}, Tx_{n(k)+1}\right) + d\left(Tx_{n(k)+1}, Tx_{n(k)}\right)$$
(8)

Taking $k \to \infty$ in the above inequalities and using (3) and (6)

$$\lim_{k\to\infty} d\left(Tx_{m(k)+1}, Tx_{n(k)+1}\right) = \beta(c) \qquad (9)$$

Now putting $x = x_{m(k)+1}$ and $y = x_{n(k)+1}$ in (1), for all $k \geq 0$, we have

$$\psi\left(d\left(Tx_{m(k)+1}, Tx_{n(k)+1}\right)\right) \leq \alpha\left(d\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right)\right) - \beta\left(d\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right)\right)$$

$$\Rightarrow \psi\left(d\left(Tx_{m(k)+1}, Tx_{n(k)+1}\right)\right) \leq \alpha\left(d\left(Tx_{mT}, Sx_{n(k)}\right)\right) - \beta\left(d\left(Tx_{m(k)}, Sx_{n(k)}\right)\right)$$

Taking $k \to \infty$ in the above inequality and using (6) and (9) with the continuity of ψ, α and β, we obtain

$$\psi(\beta(c)) \leq \alpha(\beta(c)) - \beta(\beta(c))$$

Which is a contradiction.

Therefore $\{Tx_n\}$ is a Cauchy sequence in $S(X)$ and therefore it will converge to some z, because $S(X)$ is complete. i.e.,

$$\lim_{n\to\infty} Tx_n = z. \tag{10}$$

Since $z \in S(X)$, we can find $p \in X$ such that $Sp = z$.

Now, putting $x = x_{n+1}$ and $y = p$ in (1), we have

$$\psi(d(Tx_{n+1}, Tp)) \leq \alpha(d(Sx_{n+1}, Sp)) - \beta(d(Sx_{n+1}, Sp))$$

$$\Rightarrow \psi(d(Tx_{n+1}, Tp)) \leq \alpha(d(Tx_n, z)) - \beta(d(Tx_n, z))$$

Making $n \to \infty$ in the above inequality, using (10) and the properties of ψ, α and β, we have $\psi(d(z, Tp)) \leq \theta$, which implies that $d(z, Tp) = \theta$; that is, $Tp = z$. Therefore, we have that

$$z = Tp = Sp \tag{11}$$

Hence p is a coincidence point and z is a point of coincidence of T and S. For uniqueness, suppose that there exists another point q in X such that $z_1 = Tq = Sq$ and $z \neq z_1$. Then putting $x = p$ and $y = q$ in equation (1), we have,

$$\psi(d(Tp, Tq)) \leq \alpha(d(Sp, Sq)) - \beta(d(Sp, Sq))$$

$$\Rightarrow \psi(d(z, z_1)) \leq \alpha(d(z, z_1)) - \beta(d(z, z_1))$$

Which is a contradiction unless $z = z_1$. Therefore, z is the unique point of coincidence of T and S.

Now, if T and S are weakly compatible, then by Lemma 1.5, z is the unique common fixed point of T and S.

Theorem 2.2 *Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Let $T, S : X \to X$ are mappings such that for $x, y \in X$, they satisfy the inequality*

$$\psi(d(Tx, Ty)) \leq \alpha\left(\frac{1}{2}[d(Tx, Sx) + d(Ty, Sy)]\right) - \beta\left(\frac{1}{2}[d(Tx, Sx) + d(Ty, Sy)]\right) \tag{12}$$

Where $\psi, \alpha, \beta : intP \cup \{\theta\} \to intP \cup \{\theta\}$ are such that ψ is continuous andmonotone non decreasing, α and β are continuous functions with

(i) $\psi(t) = \alpha(t) = \beta(t) = \theta$ if and only if $t = \theta$;

(ii) $\psi(t) - \alpha(t) + \beta(t) > \theta$ for all $t > \theta$;

(iii) $\beta(t) << t$ for $t \in intP$;

(iv) either $\beta(t) \leq d(x, y)$ or $d(x, y) \leq \beta(t)$ for $t \in intP \cup \{\theta\}$ and $x, y \in X$.

If $T(X) \subseteq S(x)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof: We take a sequence $\{x_n\}$ same as in the proof of Theorem 2.1. Now substituting $x = x_{n+1}$ and $y = x_{n+2}$ in equation (12), for all $n \geq 0$ we have,

$$\begin{aligned}\psi(d(Tx_{n+1}, Tx_{n+2})) &\leq \alpha\left(\frac{1}{2}[d(Tx_{n+1}, Sx_{n+1}) + d(Tx_{n+2}, Sx_{n+2})]\right) \\ &\quad - \beta\left(\frac{1}{2}[d(Tx_{n+1}, Sx_{n+1}) + d(Tx_{n+2}, Sx_{n+2})]\right)\end{aligned}$$

$$\Rightarrow \psi\big(d(Tx_{n+1},Tx_{n+2})\big) \le \alpha\left(\tfrac{1}{2}[d(Tx_{n+1},Tx_n)+d(Tx_{n+2},Tx_{n+1})]\right) - \beta\left(\tfrac{1}{2}[d(Tx_{n+1},Tx_n)+d(Tx_{n+2},Tx_{n+1})]\right) \quad (13)$$

Now, for all $n \ge 1$, we have,

$$\psi\left(\frac{1}{2}[d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})]\right) - \psi\big(d(Tx_{n+1},Tx_{n+2})\big)$$

$$\ge \psi\left(\frac{1}{2}[d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})]\right) - \alpha\left(\frac{1}{2}[d(Tx_{n+1},Tx_n)+d(Tx_{n+2},Tx_{n+1})]\right) - \beta\left(\frac{1}{2}[d(Tx_{n+1},Tx_n)+d(Tx_{n+2},Tx_{n+1})]\right)$$

$$\ge \theta$$

This implies $\psi\big(d(Tx_{n+1},Tx_{n+2})\big) \le \psi\left(\frac{1}{2}[d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})]\right)$

$$\Rightarrow d(Tx_{n+1},Tx_{n+2}) \le \frac{1}{2}[d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})]$$

$\Rightarrow d(Tx_{n+1},Tx_{n+2}) \le d(Tx_n,Tx_{n+1})$ for all $n \ge 0$.

This implies that the sequence $\{d(Tx_n,Tx_{n+1})\}$ is monotone decreasing and bounded below by θ, so it must converge because the cone P is regular. So there exist $r \in intP \cup \{\theta\}$ such that

$$d(Tx_n,Tx_{n+1}) \to r \text{ as } n \to \infty.$$

Now taking $n \to \infty$ in the equation (13) and using the continuity of ψ, α and β, we have

$$\psi(r) \le \alpha\left(\frac{1}{2}[r+r]\right) - \beta\left(\frac{1}{2}[r+r]\right)$$

$$\Rightarrow \psi(r) - \alpha(r) + \beta(r) \le \theta \Rightarrow r = \theta$$

$$\text{i.e., } \lim_{n\to\infty} d(Tx_n,Tx_{n+1}) = \theta \quad (14)$$

Now we will show that $\{Tx_n\}$ is a Cauchy sequence. Arguing like in the proof of Theorem 2.1, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ for which

$$\lim_{k\to\infty} d\big(Tx_{m(k)},Tx_{n(k)}\big) = \beta(c) \quad (15)$$

$$\text{and} \quad \lim_{k\to\infty} d\big(Tx_{m(k)+1},Tx_{n(k)+1}\big) = \beta(c) \quad (16)$$

Now putting $x = x_{m(k)+1}$and $y = x_{n(k)+1}$ in (12), for all $k \ge 0$, we have

$$\psi\left(d\big(Tx_{m(k)+1},Tx_{n(k)+1}\big)\right)$$

$$\le \alpha\left(\frac{1}{2}\left[d\big(Tx_{m(k)+1},Sx_{m(k)+1}\big)+d\big(Tx_{n(k)+1},Sx_{n(k)+1}\big)\right]\right) - \beta\left(\frac{1}{2}\left[d\big(Tx_{m(k)+1},Sx_{m(k)+1}\big)+d\big(Tx_{n(k)+1},Sx_{n(k)+1}\big)\right]\right)$$

i.e.,

$$\psi\left(d\big(Tx_{m(k)+1},Tx_{n(k)+1}\big)\right)$$

$$\le \alpha\left(\frac{1}{2}\left[d\big(Tx_{m(k)+1},Tx_{m(k)}\big)+d\big(Tx_{n(k)+1},Tx_{n(k)}\big)\right]\right) - \beta\left(\left(\frac{1}{2}\left[d\big(Tx_{m(k)+1},Tx_{m(k)}\big)+d\big(Tx_{n(k)+1},Tx_{n(k)}\big)\right]\right)\right)$$

Taking $k \to \infty$ in the above inequality and using (14) and (16) with the continuity of of $\psi, \alpha$and β, we obtain

$\psi(\beta(c)) \leq \theta \Rightarrow \beta(c) \leq \theta$, which is a contradiction.

Therefore $\{Tx_n\}$ is a Cauchy sequence in $S(X)$ and therefore it will converge to some z, because $S(X)$ is complete. i.e.,

$$\lim_{n\to\infty} Tx_n = z. \tag{17}$$

Since $z \in S(X)$, we can find $p \in X$ such that $Sp = z$.

Now, putting $x = x_{n+1}$and $y = p$ in (12), we have $\psi\left(\frac{1}{2}d(Tx_{n+1}, Tp)\right) \leq$

$$\psi(d(Tx_{n+1}, Tp)) \leq \alpha\left(\frac{1}{2}[d(Tx_{n+1}, Sx_{n+1}) + d(Tp, Sp)]\right) - \beta\left(\frac{1}{2}[d(Tx_{n+1}, Sx_{n+1}) + d(Tp, Sp)]\right)$$

$$\Rightarrow \psi\left(\frac{1}{2}d(Tx_{n+1}, Tp)\right) \leq \alpha\left(\frac{1}{2}[d(Tx_{n+1}, Tx_n) + d(Tp, z)]\right) - \beta\left(\frac{1}{2}[d(Tx_{n+1}, Tx_n) + d(Tp, z)]\right)$$

Making $n \to \infty$ in the above inequality, using (17) and properties of $\psi, \alpha$and β, we have

$$\psi\left(\frac{1}{2}d(z, Tp)\right) \leq \alpha\left(\frac{1}{2}[d(Tp, z)]\right) - \beta\left(\frac{1}{2}[d(Tp, z)]\right)$$

This implies

$$\psi\left(\frac{1}{2}d(z, Tp)\right) - \alpha\left(\frac{1}{2}[d(Tp, z)]\right) + \beta\left(\frac{1}{2}[d(Tp, z)]\right) \leq \theta$$

$\Rightarrow d(z, Tp) = \theta \Rightarrow z = Tp$, which is a contradiction unless $Tp = z$. Therefore, we have that

$$z = Tp = Sp \tag{18}$$

Hence p is a coincidence point and z is a point of coincidence of T and S.

For uniqueness, suppose that there exists another point q in X such that$z_1 = Tq = Sq$ and $z = z_1$. Then putting $x = p$ and $y = q$ in equation (12),we have,

$$\psi(d(Tp, Tq)) \leq \alpha\left(\frac{1}{2}[d(Tp, Sp) + d(Tq, Sq)]\right) - \beta\left(\frac{1}{2}[d(Tp, Sp) + d(Tp, Sq)]\right)$$

$$\Rightarrow \psi(d(z, z_1)) \leq \theta$$

which is a contradiction unless $z = z_1$. Therefore, z is the unique point of coincidence of T and S.

Now, if T and S are weakly compatible, then by Lemma 1.5, z is the unique common fixed point of T and S.

Theorem 2.3 *Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Let $T, S : X \to X$ are mappings such that for $x, y \in X$, they satisfy the inequality*

$$\psi(d(Tx, Ty)) \leq \alpha\left(\frac{1}{2}[d(Tx, Sy) + d(Ty, Sx)]\right) - \beta\left(\frac{1}{2}[d(Tx, Sy) + d(Ty, Sx)]\right) \tag{19}$$

Where $\psi, \alpha, \beta : intP \cup \{\theta\} \to intP \cup \{\theta\}$ are such that ψ is continuous andmonotone non decreasing, α and β are continuous functions with

(i) $\psi(t) = \alpha(t) = \beta(t) = \theta$ if and only if $t = \theta$;

(ii) $\psi(t) - \alpha(t) + \beta(t) > \theta$ for all $t > \theta$;

(iii) $\beta(t) << t$ for $t \in intP$;

(iv) either $\beta(t) \leq d(x, y)$ or $d(x, y) \leq \beta(t)$ for $t \in intP \cup \{\theta\}$ and $x, y \in X$.

If $T(X) \subseteq S(x)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence *in X. Moreover if T and S are weakly compatible, then T and S have a unique common fixed point in X.*

Proof: Proof is similar to that of Theorem 2.2, so it has been omitted.

Example 2.1: : Let $X = [0,1]$, $E = R^2$ with usual norm, be a real Banachspace, $P = \{(x,y) \in E : x,y \geq 0\}$ be a regular cone and the partial ordering $\leq$ with respect to the cone P, be the usual partial ordering in E.

We define $d : X \times X \to E$ as

$$d(x,y) = (|x - y|, |x - y|), \text{ for } x,y \in X.$$

Then (X,d) is a complete cone metric space with $(x,y) \in intP$, for $x,y \in X$ with $x \neq y$. Let us define $\psi, \alpha, \beta : intP \cup \{0\} \to intP \cup \{0\}$ as

$\psi(t_1,t_2) = (t_1/2, t_2/2)$, $\alpha(t_1,t_2) = (t_1/3, t_2/3)$ and $\beta(t_1,t_2) = {}^2/_3\,[(t_1^2, t_2^2)]$

for $(t_1,t_2) \in intP \cup \{0\}$. Clearly, ψ and φ has all its required properties.

Let us define $T : X \to X$ as $T(x) = \frac{x}{3} - \frac{x^2}{3}$ and $S(x) = \frac{x}{2}$, for $x \in X$. For $x,y \in X$, we can take $x > y$ without loss of generality because the equation (2) is symmetric in x and y.

Now,

$$\psi\big(d(tx,Ty)\big) = \psi\left(d\left(\frac{x}{3} - \frac{x^2}{3}, \quad \frac{y}{3} - \frac{y^2}{3}\right)\right) = \psi\left(\frac{x-y}{3} - \frac{x^2-y^2}{3}, \quad \frac{x-y}{3} - \frac{x^2-y^2}{3}\right)$$

$$= \frac{1}{2}\left(\frac{x-y}{3} - \frac{x^2-y^2}{3}, \quad \frac{x-y}{3} - \frac{x^2-y^2}{3}\right) \leq \frac{1}{2}\left(\frac{x-y}{3} - \frac{(x-y)^2}{3}, \quad \frac{x-y}{3} - \frac{(x-y)^2}{3}\right)$$

$$= \left(\frac{x-y}{6}, \quad \frac{x-y}{6}\right) - \left(\frac{(x-y)^2}{6}, \quad \frac{(x-y)^2}{6}\right)$$

$$= \alpha\left(\frac{x-y}{2}, \frac{x-y}{2}\right) - \beta\left(\frac{x-y}{2}, \frac{x-y}{2}\right) = \alpha\big(d(Sx,Sy)\big) - \beta\big(d(Sx,Sy)\big).$$

So the inequality (1) is also satisfied. Hence T and S have unique point of coincidence and also the unique common fixed point.

Here $0 \in X$ is the unique common fixed point of T and S.

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