The Generalized Difference of Biquadratic Sequence Space Γ^4

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Abstract- In this paper, we define some new biquadratic sequence spaces and obtained some topological properties of the biquadratic sequence spaces $\Gamma^4(\Delta_v^m, s, p)$, $\chi^4(\Delta_v^m, s, p)$ and $\Lambda^4(\Delta_v^m, s, p)$. We also define some inclusion relations.

Key words: Entire sequence, Analytic sequence, Gai sequence, Biquadratic sequence and Difference sequence.
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1 Introduction

ω, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences respectively. We write ω^4 for the set of all complex sequences (x_{mnkl}), where $m,n,k,l \in N$, the set of positive integers. Then, ω^4 is a linear space under the coordinatewise addition and scalar multiplication.

We can represent triple and biquadratic sequences by matrix. In the case of double sequence, we write in the form of a square. In the case of a biquadratic sequence, it will be in the form of a box in four dimensional case.

Difference sequence spaces (for single sequence) was introduced by Kizmaz [3] as follows

$$Z(\Delta) = \{x = x_k \in \omega : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞, where $(\Delta x_k) = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Later on the notation was further investigated by many others. We introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = x_{mn} \in \omega : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \Gamma^2$ and χ^2.

$$\Delta x_{mn} = x_{mn} - x_{(m+1)n} - x_{m(n+1)} + x_{(m+1)(n+1)}$$

for all $m,n \in \mathrm{N}$.

Later on, the notation was further investigated by many others. We introduce the following difference triple sequence spaces defined by [4], [7]

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$$Z(\Delta) = \{x = x_{mnk} \in \omega : (\Delta x_{mnk}) \in Z\}$$

where $Z = \Lambda^3, \Gamma^3$ and χ^3.

$$\begin{aligned} \Delta x_{mnk} = \; & x_{mnk} - x_{(m+1)nk} - x_{m(n+1)k} - x_{mn(k+1)} + x_{(m+1)(n+1)k} \\ & + x_{(m+1)n(k+1)} + x_{m(n+1)(k+1)} - x_{(m+1)(n+1)(k+1)} \end{aligned}$$

for all $m,n,k \in \mathbb{N}$.

Let ω^4, $\Lambda^4(\Delta_{mnkl}), \Gamma^4(\Delta_{mnkl})$ and $\chi^4(\Delta_{mnkl})$ denote the space of all complex sequences, biquadratic analytic difference sequence space, biquadratic entire difference sequence space and biquadratic chi difference sequence space respectively and is defined as

$$Z(\Delta) = \{x = x_{mnkl} \in \omega : (\Delta x_{mnkl}) \in Z\}$$

where $Z = \Lambda^4, \Gamma^4$ and χ^4

$$\begin{aligned} \Delta(x_{mnkl}) = \; & x_{mnkl} - x_{(m+1)nkl} - x_{m(n+1)kl} - x_{mn(k+1)l} - x_{mnk(l+1)} \\ & + x_{(m+1)(n+1)kl} + x_{(m+1)n(k+1)l} + x_{(m+1)nk(l+1)} \\ & + x_{m(n+1)(k+1)l} + x_{m(n+1)k(l+1)} + x_{mn(k+1)(l+1)} \\ & - x_{(m+1)(n+1)(k+1)l} - x_{(m+1)(n+1)k(l+1)} \\ & - x_{m(n+1)(k+1)(l+1)} - x_{(m+1)n(k+1)(l+1)} \\ & + x_{(m+1)(n+1)(k+1)(l+1)} \end{aligned}$$

for all $m, n, k, l \in \mathbb{N}$.

$$\begin{aligned} \Delta^m x_{mnkl} = \; & \Delta\Delta^{m-1} x_{mnkl} \\ = \; & \Delta^{m-1} x_{mnkl} - \Delta^{m-1} x_{(m+1)nkl} - \Delta^{m-1} x_{m(n+1)kl} \\ & - \Delta^{m-1} x_{mn(k+1)l} - \Delta^{m-1} x_{mnk(l+1)} + \Delta^{m-1} x_{(m+1)(n+1)kl} \\ & + \Delta^{m-1} x_{(m+1)n(k+1)l} + \Delta^{m-1} x_{(m+1)nk(l+1)} + \Delta^{m-1} x_{m(n+1)k(l+1)} \\ & + \Delta^{m-1} x_{m(n+1)(k+1)l} + \Delta^{m-1} x_{mn(k+1)(l+1)} - \Delta^{m-1} x_{(m+1)(n+1)(k+1)l} \\ & - \Delta^{m-1} x_{m(n+1)(k+1)(l+1)} - \Delta^{m-1} x_{(m+1)n(k+1)(l+1)} \\ & - \Delta^{m-1} x_{(m+1)(n+1)k(l+1)} + \Delta^{m-1} x_{(m+1)(n+1)(k+1)(l+1)} \end{aligned}$$

2 Preliminaries

Let ω^4 denote the set of all complex sequences (x_{mnkl}) where $m,n,k,l \in \mathbb{N}$. A sequence $x = (x_{mnkl})$ is said to be biquadratic analytic sequence if

$$\sup_{m,n,k,l} |x_{mnkl}|^{\frac{1}{m+n+k+l}} < \infty$$

The vector space of all biquadratic analytic sequences will be denoted by Λ^4.

A sequence $x = (x_{mnkl})$ is called biquadratic entire sequence if

$$|x_{mnkl}|^{\frac{1}{m+n+k+l}} \to 0 \qquad \text{as } m,n,k,l \to \infty.$$

The set of all biquadratic entire sequences will be denoted by Γ^4.

A sequence $x = (x_{mnkl})$ is called biquadratic chi sequence if

$$(m+n+k+l)!\,|x_{mnkl}|^{\frac{1}{m+n+k+l}} \to 0 \text{ as } m,n,k,l \to \infty$$

The vector space of all biquadratic analytic sequences will be denoted by χ^4.

Throughout the article ω^4, $\Lambda^4(\Delta_{mnkl})$,$\Gamma^4(\Delta_{mnkl})$ denote the spaces of all complex sequence spaces, biquadratic entire difference sequence spaces and biquadratic analytic difference sequence spaces respectively. For a biquadratic sequence $x \in \omega^4$,we define the sets

$$\Gamma^4(\Delta_{mnkl}) = \left\{x \in \omega^4: |\Delta x_{mnkl}|^{\frac{1}{m+n+k+l}} \to 0 \text{ as } m,n,k,l \to \infty\right\}$$

$$\Lambda^4(\Delta_{mnkl}) = \left\{x \in \omega^4: \sup_{m,n,k,l} |\Delta x_{mnkl}|^{\frac{1}{m+n+k+l}} < \infty\right\}$$

$$\chi^4(\Delta_{mnkl}) = \{x \in \omega^4: \sup_{m,n,k,l} \left((m+n+k+l)!\, |\Delta x_{mnkl}|\right)^{\frac{1}{m+n+k+l}} \to 0 \text{ as } m,n,k,l \to \infty\}$$

The spaces $\Lambda^4(\Delta_{mnkl})$ and $\Gamma^4(\Delta_{mnkl})$ are metric space with the metric

$$d(x,y) = \sup_{m,n,k,l} \left\{|\Delta x_{mnkl} - \Delta y_{mnkl}|^{\frac{1}{m+n+k+l}} : m,n,k,l = 1,2,3,\ldots\right\} \qquad (2.1)$$

for all $x = (x_{mnkl})$ and $y = (y_{mnkl})$ in $\Lambda^4(\Delta)$ and $\Gamma^4(\Delta)$.

The space $\chi^4(\Delta_{mnkl})$ is a metric space with the metric

$$d(x,y) = \sup_{m,n,k,l} \left\{\left((m+n+k+l)!\, |\Delta x_{mnkl} - \Delta y_{mnkl}|\right)^{\frac{1}{m+n+k+l}} : m,n,k,l = 1,2,3,\ldots\right\} \qquad (2.2)$$

for all $x = (x_{mnkl})$ and $y = (y_{mnkl})$ in $\chi^4(\Delta)$.
Now we define the following sequence spaces, Let $s \geq 0$ be real number and $v = (v_{mnkl})$ be non-zero sequence, then

$$\Gamma^4(\Delta_v^m, s, p) = \{x = (x_{mnkl}) \in \omega^4: (mnkl)^{-s}\left(|\Delta_v^m x_{mnkl}|^{\frac{1}{m+n+k+l}}\right)^{p_{mnkl}} \to 0 \text{ as } m,n,k,l \to \infty\ s \geq 0\}$$

$$\Lambda^4(\Delta_v^m, s, p) = \{x = (x_{mnkl}) \in \omega^4: (mnkl)^{-s} \sup_{m,n,k,l} \left(|\Delta_v^m x_{mnkl}|^{\frac{1}{m+n+k+l}}\right)^{p_{mnkl}} < \infty,\ s \geq 0\}$$

and

$$\chi^4(\Delta_v^m, s, p) = \{x = (x_{mnkl}) \in \omega^4: (mnkl)^{-s} \sup_{mnkl} (((m+n+k+l)!\, |\Delta_v^m x_{mnkl}|)^{\frac{1}{m+n+k+l}})^{p_{mnkl}} \to 0 \text{ as } m,n,k,l \to \infty\ s \geq 0\}$$

$$\begin{aligned}
\Delta_v^0 x_{mnkl} &= (v_{mnkl}x_{mnkl})\,\Delta_v x_{mnkl} \\
&= v_{mnkl}x_{mnkl} - v_{(m+1)nkl}x_{(m+1)nkl} - v_{m(n+1)kl}x_{m(n+1)kl} \\
&\quad - v_{mn(k+1)l}x_{mn(k+1)l} - v_{mnk(l+1)}x_{mnk(l+1)} + v_{(m+1)(n+1)kl}x_{(m+1)(n+1)kl} \\
&\quad + v_{(m+1)n(k+1)l}x_{(m+1)n(k+1)l} + v_{(m+1)nk(l+1)}x_{(m+1)nk(l+1)} \\
&\quad + v_{m(n+1)(k+1)l}x_{m(n+1)(k+1)l} + v_{m(n+1)k(l+1)}x_{m(n+1)k(l+1)} \\
&\quad + v_{mn(k+1)(l+1)}x_{mn(k+1)(l+1)} - v_{(m+1)(n+1)(k+1)l}x_{(m+1)(n+1)(k+1)l} \\
&\quad - v_{(m+1)(n+1)k(l+1)}x_{(m+1)(n+1)k(l+1)} - v_{(m+1)n(k+1)(l+1)}x_{(m+1)n(k+1)(l+1)} \\
&\quad - v_{m(n+1)(k+1)(l+1)}x_{m(n+1)(k+1)(l+1)} \\
&\quad + v_{(m+1)(n+1)(k+1)(l+1)}x_{(m+1)(n+1)(k+1)(l+1)}
\end{aligned}$$

$$\begin{aligned}
\Delta_v^m x_{mnkl} &= \Delta\Delta_v^{m-1} x_{mnkl} \\
&= \Delta_v^{m-1}x_{mnkl} - \Delta_v^{m-1}x_{(m+1)nkl} - \Delta_v^{m-1}x_{m(n+1)kl} - \Delta_v^{m-1}x_{mn(k+1)l} \\
&\quad - \Delta_v^{m-1}x_{mnk(l+1)} + \Delta_v^{m-1}x_{(m+1)(n+1)kl} + \Delta_v^{m-1}x_{(m+1)n(k+1)l} \\
&\quad + \Delta_v^{m-1}x_{(m+1)nk(l+1)} + \Delta_v^{m-1}x_{m(n+1)(k+1)l} + \Delta_v^{m-1}x_{mn(k+1)(l+1)} \\
&\quad + \Delta_v^{m-1}x_{m(n+1)k(l+1)} - \Delta_v^{m-1}x_{m(n+1)(k+1)(l+1)} \\
&\quad - \Delta_v^{m-1}x_{(m+1)n(k+1)(l+1)} - \Delta_v^{m-1}x_{(m+1)(n+1)(k+1)l} \\
&\quad - \Delta_v^{m-1}x_{(m+1)(n+1)k(l+1)} + \Delta_v^{m-1}x_{(m+1)(n+1)(k+1)(l+1)}
\end{aligned}$$

we get the following sequence spaces from the above sequence spaces by choosing some special p,m,s and v .If $s = 0, m = 1$ and

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

with 1 upto $(m,n,k,l)^{th}$ position and zero otherwise and $p_{mnkl} = 1$ for all m,n,k,l. We have

$$\Gamma^4 = \{x = (x_{mnkl}) : \Delta x \in \Gamma^4\}$$

$$\Lambda^4 = \{x = (x_{mnkl}) : \Delta x \in \Lambda^4\}$$

$$\chi^4 = \{x = (x_{mnkl}) : \Delta x \in \chi^4\}$$

If $s = 0$ and $p_{mnkl} = 1$ for all m,n,k,l we have the following sequence spaces

$$\Gamma^4 = \{x = (x_{mnkl}) : \Delta x \in \Gamma^4\}$$

$$\Lambda^4 = \{x = (x_{mnkl}) : \Delta x \in \Lambda^4\}$$

$$\chi^4 = \{x = (x_{mnkl}) : \Delta x \in \chi^4\}$$

If $s = 0, m = 0$ and

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

with 1 upto $(m,n,k,l)^{th}$ position and zero otherwise. We have the following sequence spaces

$$\Gamma^4(p) = \{x = (x_{mnkl}) \in \omega^4 : |\Delta x_{mnkl}|^{\frac{1}{m+n+k+l}} \to 0 \text{ as } m,n,k,l \to \infty\}$$

$$\Lambda^4(p) = \{x = (x_{mnkl}) \in \omega^4 : \sup_{m,n,k,l} |\Delta x_{mnkl}|^{\frac{1}{m+n+k+l}} < \infty\}$$

$$\chi^4(p) = \{x = (x_{mnkl}) \in \omega^4 : \sup_{m,n,k,l} (((m+n+k+l)!\, |\Delta x_{mnkl}|^{\frac{1}{m+n+k+l}} < \infty\}$$

If $m = 0$ and

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

with 1 upto $(m,n,k,l)^{th}$ position and zero otherwise. We have the following sequence spaces

$$\Gamma^4(p,s) = \{x = (x_{mnkl}) \in \omega^4 : (mnkl)^{-s} |\Delta x_{mnkl}|^{\frac{p_{mnkl}}{m+n+k+l}} \to 0 \text{ as } m,n,k,l \to \infty,\ s \geq 0\}$$

$$\Lambda^4(p,s) = \{x = (x_{mnkl}) \in \omega^4 : \sup_{m,n,k,l} (mnkl)^{-s} |\Delta x_{mnkl}|^{\frac{p_{mnkl}}{m+n+k+l}} < \infty,\ s \geq 0\}$$

$$\chi^4(p,s) = \{x = (x_{mnkl}) \in \omega^4 : \sup_{m,n,k,l} (mnkl)^{-s} (m+n+k+l)!\, |\Delta x_{mnkl}|^{\frac{p_{mnkl}}{m+n+k+l}} < \infty,\ s \geq 0\}$$

If $s = 0, m = 0$ and $p_{mnkl} = 1$

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

for all m, n, k, l with 1 upto $(m, n, k, l)^{th}$ position and zero otherwise. We have Γ^4, χ^4 and Λ^4.
If $s = 0$ we have $\Gamma^4(\Delta_v^m, p)$, $\Lambda^4(\Delta_v^m, p)$ and $\chi^4(\Delta_v^m, p)$.
For a subspace ψ of a linear space is said to be sequence algebra if $x, y \in \psi$ implies that $x.y = (x_{mnkl} y_{mnkl}) \in \psi$.

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

for all m, n, k, l with 1 upto $(m, n, k, l)^{th}$ position and zero otherwise. We have Γ^4, χ^4 and Λ^4.
If $s = 0$ we have $\Gamma^4(\Delta_v^m, p)$, $\Lambda^4(\Delta_v^m, p)$ and $\chi^4(\Delta_v^m, p)$.
For a subspace ψ of a linear space is said to be sequence algebra if $x, y \in \psi$ implies that $x.y = (x_{mnkl}y_{mnkl}) \in \psi$,

A sequence E is said to be solid (or normal) if $(\lambda_{mnkl}x_{mnkl}) \in E$, whenever $(x_{mnkl}) \in E$ or all sequences of scalars $(\lambda_{mnkl} = k)$ with $|\lambda_{mnkl}| \leq 1$.
If X is a linear space over the field $\mathbb{C}$, then a paranorm on X is a function $g: g(\theta) = 0$ where

$$\begin{aligned} \theta &= (0,0,0, \ldots), \\ g(-x) &= g(x), \\ g(x+y) &\leq g(x) + g(y) \end{aligned}$$

And $$|\lambda - \lambda_0| \to 0,$$

$$g(x - x_0) \Rightarrow g(\lambda x - \lambda_0 x_0) \to 0,$$

where $\lambda, \lambda_0 \in \mathbb{C}$ and $x, x_0 \in X$. A paranormed space is a linear space X with a paranorm g and is written (X, g).
In this paper , we define some new sequence spaces and give some topological properties of the sequence spaces

$$\Gamma^4(\Delta_v^m, s, p), \Lambda^4(\Delta_v^m, s, p)$$

and

$$\chi^4(\Delta_v^m, s, p)$$

And investigate some inclusion relations.

3. Main Results

Theorem 3.1: *The following statements hold*
(i) $\Gamma^4(\Delta_v^m, s) \subset \Lambda^4(\Delta_v^m, s)$ *and the inclusion is strict.*
(ii $X(\Delta_v^m, s, p) \subset X(\Delta_v^{m+1}, s, p)$ *does not hold for any* $X = \Gamma^4$, Λ^4 and χ^4.

Proof:(i) If we choose $s = 0$,

$$x = \begin{bmatrix} 1 & 0 & \ldots 1 & 1 & 0\ldots \\ 1 & 0 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 0 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

And

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

Hence $x \in \Lambda^4(\Delta_v^m, s)$ but $x \notin \Gamma^4(\Delta_v^m, s)$
(ii) Let

$$v = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ & & & & . \\ & & & & . \\ & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}$$

$p = (p_{mnkl})$ and $x = (x_{mnkl})$ given by

$$p_{mnkl} = 1, |x_{mnkl}|^{\frac{1}{m+n+k+l}} = m^2n^2k^2l^2$$

if m,n,k,l is odd

$$p_{mnkl} = 2, |x_{mnkl}|^{\frac{1}{m+n+k+l}} = mnkl$$

if m,n,k,l is even 0 otherwise.

Since for $m,n,k,l \geq 1$

$$|(\Delta_v^0 x_{mnkl})|^{\frac{p_{mnkl}}{m+n+k+l}} = |x_{mnkl})|^{\frac{p_{mnkl}}{m+n+k+l}} = m^2n^2k^2l^2$$

$$m^{-3}n^{-3}k^{-3}l^{-3}|(\Delta_v^0 x_{mnkl})|^{\frac{p_{mnkl}}{m+n+k+l}} = m^{-3}n^{-3}k^{-3}l^{-3}m^2n^2k^2l^2$$
$$= m^{-1}n^{-1}k^{-1}l^{-1} \to 0 (m,n,k,l \to \infty)$$

and for $j \geq 1$

$$\left|\Delta_v x_{2j,2j,2j,2j}\right|^{p_{2j,2j,2j,2j}/8j} = (8j^3 + 8j^2 + 1)^2$$
$$\left|\Delta_v x_{2j,2j,2j,2j}\right|^{p_{2j,2j,2j,2j}/8j} \geq 8j \to 0$$

Now, we can see that $x \in \Gamma^4(\Delta_v^m, 3, p)$ and $x \notin \Lambda^4(\Delta_v^m, 3, p)$, which implies that $X(\Delta_v^m, s, p)$ is not subset of $X(\Delta_v^{m+1}, s, p)$. This completes the proof.

Theorem 3.2:

(i) $\Gamma^4(\Delta_v^m, s)$and$\Lambda^4(\Delta_v^m, s)$ *are linear spaces over the field* R.
(ii) $\chi^4(\Delta_v^m, s)$and $\Lambda^4(\Delta_v^m, s)$ *are linear spaces over the field* R.

Proof: (i) Suppose that

$M = \max\{1, \sup_{m,n,k,l \geq N} p_{mnkl}\}$Since $p_{mnkl}/M \leq 1$, We have for m, n, k, l

$$|\Delta_v^m (x_{mnkl} + y_{mnkl})|^{p_{mnkl}/M} \leq |\Delta_v^m (x_{mnkl})|^{p_{mnkl}/M} + |\Delta_v^m (x_{mnkl})|^{p_{mnkl}/M} \quad \textbf{(3.1)}$$

and for all $\lambda \in R$

$$|\lambda|^{p_{mnkl}/M} \leq max\{1, |\lambda|\} \quad \textbf{(3.2)}$$

Now the linearity follows from (3.1) and (3.2).

(ii) We can prove this same as (i)

Theorem3.3:

$$N_1 = \min\left\{n_0: \sup_{m,n,k,l\geq n_0} (mnkl)^{-s}\left(|\Delta_v^m x_{mnkl}|^{1/m+n+k+l}\right)^{p_{mnkl}} \infty\right\}$$

$$N_2 = \min\left\{n_0: \sup_{m,n,k,l\geq n_0} p_{mnkl} \leq \infty\right\}$$

and

$$N = \max\{N_1, N_2\}$$

$\Gamma^4(\Delta_v^m, s, p)$ is a paranorm space with

$$g(x) = \sum_{m,n,k,l=1}^{p,q,r,s} |x_{mnkl}| + \lim_{n\to\infty} \sup_{m,n,k,l\geq N} (mnkl)^{-s}|\Delta_v^m x_{mnkl}|^{p_{mnkl}/M}$$

if and only if $\mu > 0$, where $\mu = \lim_{N\to\infty} \inf_{m,n,k,l\geq N} p_{mnkl}$ and

$$M = max\left\{1, \sup_{m,n,k,l\geq N} p_{mnkl}\right\}$$

Proof: Let $\Gamma^4(\Delta_v^m, s, p)$ be a paranorm space with $g(x)$ and suppose that $\mu = 0$. Then $\alpha = \inf_{m,n,k,l\geq N} p_{mnkl} = 0$ for all $N \in$ N and hence we obtain-

$$g(x) = \sum_{m,n,k,l=1}^{p,q,r,s} |x_{mnkl}| + \lim_{n\to\infty} \sup_{m,n,k,l\geq N} (mnkl)^{-s}|\Delta_v^m x_{mnkl}|^{p_{mnkl}/M}$$

for all $\lambda \in (0,1]$, where $x = \alpha \in \Gamma^4(\Delta_v^m, s, p)$ when $\lambda \to 0$ imply $\lambda x \to 0$, when x is fixed. But this contradicts to (4.3.3) to be a paranorm.

Sufficient- Let $\mu > 0$ it is trivial that $g(0) = 0$, $g(-x) = g(x)$ and $g(x+y) = g(x) + g(y)$

Since $\mu > 0$∃ a positive number α such that $p_{mnkl} > \alpha$ for sufficiently large positive integer m,n,k,l. Hence for any $\lambda \in$ R, we may write-

$$|\lambda|^{p_{mnkl}} \leq \max(|\lambda|^M, |\lambda|^\alpha)$$

for sufficiently large positive integer m,n,k,l. Therefore, we obtain that-

$$g(\lambda x) \leq \max(|\lambda|^M, |\lambda|^\alpha) g(x)$$

using this, one can prove that $\lambda x \to 0$, whenever x is fixed and $\lambda \to 0$.

Theorem 3.4: *Let* $0 < p_{mnkl} \leq q_{mnkl} \leq 1$ for $m,n,k,l \in$ N, *then*

(i) $\chi^4(\Delta^m{}_v, s, p) \subseteq \chi^4(\Delta^m{}_v, s, q)$

(ii) $\Gamma^4(\Delta^m{}_v, s, p) \subseteq \Gamma^4(\Delta^m{}_v, s, q)$

(iii) $\Lambda^4(\Delta^m{}_v, s, p) \subseteq \Lambda^4(\Delta^m{}_v, s, q)$

Proof: Let $x \in \Lambda^4(\Delta_v^m, s, p)$. Then ∃ a constant $M > 1$ such that

$$(mnkl)^{-s}|\Delta_v^m x_{mnkl}|^{\frac{p_{mnkl}}{m+n+k+l}} \leq M,$$

for all m, n, k, l.
Suppose that $x^i \in \Lambda^4(\Delta_v^m, s, q)$ and $x^i \to x \in \Lambda^4(\Delta_v^m, s, p)$.
Then for every $0 < \epsilon < 1$, $\exists\, N$ such that for all m, n, k, l.

$$(mnkl)^{-s}\left|\Delta_v^m\left(x_{mnkl}^i - x_{mnkl}\right)\right|^{\frac{p_{mnkl}}{m+n+k+l}} < \epsilon \text{ for all } \quad i > N$$

Now,

$$\begin{aligned}(mnkl)^{-s}| \quad & \Delta_v^m\left(x_{mnkl}^i - x_{mnkl}\right)|^{\frac{q_{mnkl}}{m+n+k+l}} \\ & < (mnkl)^{-s}\left|\Delta_v^m\left(x_{mnkl}^i - x_{mnkl}\right)\right|^{\frac{p_{mnkl}}{m+n+k+l}} \\ & < \epsilon \text{ for all } \quad i > N\end{aligned}$$

$$x \in \Lambda(\Delta_v^m, s, q).$$

(ii) It is easy. Therefore we omit the proof.

Theorem 3.5: *For $X = \Gamma, \Lambda$ and χ then we obtain*

(i) $X(\Delta_v^m, s, p)$ *is not sequence algebra.*

(ii) $X(\Delta_v^m, s, p)$ *is not solid.*

Proof: Example-

$$p_{mnkl} = 1,\ v_{mnkl} = \frac{1}{(mnkl)^{2(m+n+k+l)}},$$

and

$$x_{mnkl} = (mnkl)^{2(m+n+k+l)},\ y_{mnkl} = (mnkl)^{2(m+n+k+l)}$$

Then we have

$$x, y \in \Gamma^4(\Delta_v^m, 0, p) \text{ but } x, y \in \Gamma^4(\Delta_v^m, 0, p) \text{ with } m = 1,\ s = 0.$$

Example-

$$x_{mnkl} = \begin{bmatrix} 1 & 1 & \ldots 1 & 1 & 0, & \ldots \\ 1 & 1 & \ldots 1 & 1 & 0, & \ldots \\ & & & & & . \\ & & & & & . \\ & & & & & . \\ 1 & 1 & \ldots 1 & 1 & 0, & \ldots \\ 0 & 0 & \ldots 0 & 0 & 0, & \ldots \end{bmatrix}$$

Let

$$p_{mnkl} = 1, \alpha_{mnkl} = (-1)^{m+n+k+l}$$

then

$$\alpha_{mnkl} x_{mnkl} \notin \Gamma^4(\Delta_v^m, s, p)$$

with $m = 1$ and $s = 0$.

The following proposition's proof is routine verification.

.

Theorem 3.6: *For $X = \Gamma, \Lambda$ and χ then we obtain*

(i) $s_1 < s_2 \Rightarrow X^4(\Delta_v^m, s_1, p) \subset X(\Delta_v^m, s_2, p)$

(ii) Let $0 < \inf p_{mnkl} < p_{mnkl} < 1$ then $X(\Delta_v^m, s, p) \subset X(\Delta_v^m, s)$

(iii) Let $0 \leq p_{mnkl} \leq \sup_{m,n,k,l} < \infty$ then $X(\Delta_v^m, s) \subset X(\Delta_v^m, s, p)$

(iv) Let $0 \leq p_{mnkl} \leq q_{mnkl}$ and $\left(\frac{q_{mnkl}}{p_{mnkl}}\right)$, $X(\Delta_v^m, s, q) \subset X(\Delta_v^m, s, p)$.

4. Conclusion

In present chapter, we defined Biquadratic analytic, biquadratic entire and biquadratic gai difference sequence spaces. We introduced metric on each which makes them metric spaces and also obtained some new sequence spaces $\Gamma^4(\Delta_v^m, s, p)$, $\lambda^4(\Delta_v^m, s, p)$ and $\chi^4(\Delta_v^m, s, p)$ with their topological properties. Main point of the chapter is the inclusion relations among defined difference sequence spaces and now these spaces become paranormed spaces. In the concluding point we give examples, why these spaces are not sequence algebra and solid.

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