CONCENTRATION PROBABILITIES OF THE GENERALIZED QUASI MINIMAX AND MOCK-MINIMAX ESTIMATORS IN LINEAR REGRESSION MODEL

Dr. SYED QAIM AKBAR RIZVI

Department of Statistics

SHIA P.G. COLLEGE,

LUCKNOW UNIVERSITY, LUCKNOW.

ABSTRACT

The generalized quasi minimax and mock-minimax estimators for the estimation of the coefficient vector ß of regression coefficients and judging their performance under a decision theoretic frame work based on their quadratic risks. For judging the performance of an estimator, Rao (1981) suggested the criteria such as Pitman nearness and concentration probability of an estimator around the true parametric value under which he showed empirically that some well known minimax mean squared error estimators do not work satisfactorily. For further details regarding the justifications of the use of criteria other than minimax mean squared error criterion, see Rao (1981). Keeping in mind, the justifications given by Rao (1981)

2. The estimators and the regression model

Let the linear regression model be

$$Y = X\beta + u$$

Where Y is a Tx1 vector of observations on the variable to be explained, β is a px1 vector of regression coefficients to be estimated, X is a non-stochastic Txp full column rank matrix of observations on p explanatory variables and u is a Tx1 vector of disturbances following multivariate normal distribution with

$$E(u) = 0$$

$$E(uu') = \sigma^2 I_T$$

In which σ^2 is the variance of disturbances.

Under ellipsoidal constraints

$$B = (\beta : \beta' H\beta \leq 1) \quad \text{------------------------(2)}$$

Where H is a known positive definite symmetric matrix, trenkler and stahlecker (1984) gave the quasi minimax estimator

$$\hat{\beta}_1 = (X'X + \sigma^2 H)^{-1} X'Y \quad \text{------------------------(3)}$$

When σ^2 is know.

Subject to the constraints (2), the mock-minimax estimator is given by

$$\hat{\beta}_2 = \left\{1 - \frac{(\sigma^2 \ tr \ A(X'X)^{-1})}{\overline{Ch}(AH^{-1}) + \sigma^2 \ tr \ A(X'X)^{-1}}\right\} b \quad \text{------------------------(4)}$$

When, σ^2 is not known, the adaptive versions of the quasi minimax estimator $\hat{\beta}_1$ and the mock-minimax estimator $\hat{\beta}_2$ are

$$b_1 = (X'X + \hat{\sigma}^2 H)^{-1} X'Y$$

And

$$b_2 = \left\{1 - \frac{\hat{\sigma}^2 \ tr \ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1}) + \ \hat{\sigma}^2 \ tr \ A(X'X)^{-1}}\right\} b$$

Where

$$\hat{\sigma}^2 = \delta\sigma_0^2 + (1-\delta)s^2$$

Generalized versions of b_1 and b_2 known as the generalized quasi minimax and the generalized mock minimax estimators are

$$b_1^* = \left(X'X + \hat{\sigma}_g^2 H\right)^{-1} X'y$$

And

$$b_2^* = \left\{1 - \frac{\hat{\sigma}_g^2 \ tr \ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1}) + \hat{\sigma}_g^2 \ tr \ A(X'X)^{-1}}\right\} b$$

We now find the concentration probabilities of $b_1^* \ and \ b_2^*$ the estimators and in next section 3.

3. CONCENTRATION PROBABILITIES OF ESTIMATORS b_1^* and b_2^*

We first introduce the following notations:

$$r_1 = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b_1^* - \beta)$$

$$\xi(r_1) = \frac{1}{(2\pi)^{\frac{p}{2}}}\ e^{-\frac{1}{2}r_1' r_1}$$

And

$$r_2 = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b_2^* - \beta)$$

$$\xi(r_2) = \frac{1}{(2\pi)^{\frac{p}{2}}}\ e^{-\frac{1}{2}r_2' r_2}$$

$$(b_1^* - \beta) = \emptyset_{-\frac{1}{2}} + \emptyset_{-1} + \emptyset_{-\frac{3}{2}} + \emptyset_{-2} + 0\left(T^{-\frac{5}{2}}\right)$$

where

$$\emptyset_{-\frac{1}{2}} = (X'X)^{-1}X'u$$

$$\emptyset_{-1} = -\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}(X'X)^{-1}H\beta$$

$$\emptyset_{-\frac{3}{2}} = -\left(1 - g'(1)\right)\epsilon\,(X'X)^{-1}H\beta$$

$$-\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}(X'X)^{-1}H(X'X)^{-1}X'u$$

$$\emptyset_{-2} = -\left(1 - g'(1)\right)\epsilon\,(X'X)^{-1}H(X'X)^{-1}X'u$$

$$+\frac{\left(1 - g'(1)\right)}{T(T-p)}\ (u'(X'X)u)(X'X)^{-1}H\beta$$

$$+\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}^2(X'X)^{-1}H(X'X)^{-1}H\beta$$

$$+\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}^2(X'X)^{-1}H(X'X)^{-1}H\beta$$

Where $\epsilon = (s^2 - \sigma^2)$ is an stochastic quantity of order

$$O_p\left(T^{-\frac{1}{2}}\right)\ with\ E(\epsilon) = 0\ and\ E(\epsilon^2) = \frac{2}{T}\sigma^4$$

Defining $z = \frac{1}{\sigma}(X'X)^{-\frac{1}{2}}X'u$, we observe that z follows multivariate normal distribution with mean vector zero and dispersion matrix I_P.

Further, $\left(\frac{u'\bar{P}_X u}{\sigma^2}\right)$ follows a chi-square distribution with $n = T - p$ degrees of freedom and Z and $\left(\frac{u'\bar{P}_X u}{\sigma^2}\right)$ are independently distributed.

From (3,3,2) , we have

$$r_1 = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b_1^* - \beta)$$

$$= \tau_0 + \tau_{-\frac{1}{2}} + \tau_{-\frac{3}{2}} + O_p(T^{-2})$$

where

$$\tau_0 = Z$$

$$\tau_{-\frac{1}{2}} = -\frac{\lambda}{\sigma}\alpha_1$$

$$\tau_{-1} = -\frac{(1 - g'(1))}{\sigma}\epsilon\,\alpha_1 - \lambda CZ$$

$$\tau_{-\frac{3}{2}} = -(1 - g'(1))\epsilon\, CZ + \frac{\sigma(1 - g'(1))}{nT}(Z'X'XZ)\alpha_1 + \frac{\lambda^2}{\sigma}C\alpha_1$$

Where $\lambda = g'(1)\sigma_o^2 + (1 - g'(1))\sigma^2$

The approximate characteristic function of vector r_1 upto order $\left(T^{-\frac{3}{2}}\right)$, can then be obtained as follows

$$\zeta_{r_1}(h) = E\left(e^{ih'r_1}\right)$$

$$= E\left\{e^{ih\zeta_0}\, e^{\left(ih'\zeta_{-\frac{1}{2}} + ih'\zeta_{-1} + ih'\zeta_{-\frac{3}{2}}\right)}\right\}$$

$$= E\left[e^{ih\zeta_0}\left\{1+\left(ih'\zeta_{-\frac{1}{2}}\right)+ih'\zeta_{-1}+\frac{1}{2}\left(ih'\zeta_{-\frac{1}{2}}\right)^2+\left(ih'\zeta_{-\frac{3}{2}}\right)\right.\right.$$
$$\left.\left.+\left(ih'\zeta_{-\frac{1}{2}}\right)(ih'\zeta_{-1})+\frac{1}{6}\left(ih'\zeta_{-\frac{1}{2}}\right)^3\right\}\right]$$

Where h is a px1 colomn vector of fixed constants. Further we observe that for a fixed px1 vector a and pxp matrix A

$$E\left(e^{(ih'Z)}\right)=e^{-\frac{1}{2}h'h}$$

$$E(\alpha'Z)e^{ih'Z}=i(\alpha'h)\,e^{-\frac{1}{2}h'h}$$

$$E(Z'AZ)e^{ih'Z}=(tr\,A-h'Ah)e^{-\frac{1}{2}h'h}$$

With the help of results along with, the characteristic function of the vector r_1, upto order $O_p\left(T^{-\frac{3}{2}}\right)$ has been obtained as

$$\zeta_{r1}(h)=\left(1+\zeta_{-\frac{1}{2}}+\zeta_{-1}+\zeta_{-\frac{3}{2}}\right)e^{-\frac{1}{2}h'h}$$

Where

$$\zeta_{-\frac{1}{2}}=\frac{i\lambda}{\sigma}(h'\alpha_1)^2$$

$$\zeta_{-1}=\lambda(h'ch)-\frac{2\lambda}{2\sigma^2}(h'\alpha_1)^2$$

$$\zeta_{-\frac{3}{2}}=\frac{i\left(1-g'(1)\right)}{nT}(trX'X-h'X'Xh)(h'\alpha_1)+\frac{i\lambda^2}{\sigma}(h'c\alpha_1)$$
$$-\frac{i\lambda^2}{\sigma}\left((h'\alpha_1)h'ch\right)+\frac{i\lambda^3}{6\sigma^3}(h'\alpha_1)^3$$

Utilizing the following results for a fixed px1 vector α and a pxp matrix A,

$$\frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\dots\dots\dots\dots\int_{-\infty}^{\infty}e^{-ih'r_1-\frac{1}{2}h'h}\;dh=\xi(r_1)$$

$$\frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\dots\dots\dots\int_{-\infty}^{\infty}(\alpha'h)\,e^{-ih'r_1-\frac{1}{2}h'h}\;dh=-i(\alpha'r_1)\xi(r_1)$$

$$\frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\dots\dots\dots\int_{-\infty}^{\infty}(h'Ah)\, e^{-ih'r_1-\frac{1}{2}h'h}\ dh = (tr\ A - r_1'Ar_1)\xi(r_1)$$

$$\frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\dots\dots\dots\int_{-\infty}^{\infty}(\alpha'h)\,(h'Ah)e^{-ih'r_1-\frac{1}{2}h'h}\ dh$$

$$= i(\alpha'r_1)(r_1'Ar_1) - (\alpha'r_1)\ tr\ A - \alpha'(A+A')r_1$$

Along with the inversion theorem

$$f(r_1) = \frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\dots\dots\dots\int_{-\infty}^{\infty} e^{-ih'r_1}\,\xi_{r_1}(h)\, dh,$$

Where f(r_1) is the probability density function of the random vector r_1, we get the large sample asymptotic approximation for the joint probability density function of the elements of vector r_1 upto order $0_p\left(T^{-\frac{3}{2}}\right)$ to be

$$f(r_1) = \left\{1 + Y_{-\frac{1}{2}} + Y_{-1} + Y_{-\frac{3}{2}}\right\}\xi(r_1)$$

Where

$$Y_{-\frac{1}{2}} = -\frac{1}{\sigma}\{g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\}\alpha_1'r_1$$

$$Y_{-1} = \{g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\}(tr\ C - r_1'Cr_1)$$
$$-\frac{1}{2\sigma^2}\{g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\}^2(\alpha_1'\alpha_1 - (\alpha_1'\alpha_1)^2)$$

$$Y_{-\frac{3}{2}} = \frac{\left(1-g'(1)\right)}{nT}\ (\alpha_1'r_1)(r_1'X'Xr_1) - 2(\alpha_1'X'Xr_1)$$
$$+\frac{1}{\sigma}\{g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\}^2\ \{(\alpha_1'r_1)(r_1'Cr_1)\ - (\alpha_1'r_1)\ tr\ C$$
$$-(\alpha_1'Cr_1)\} - \frac{1}{6\sigma^3}\{g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\}^3(\alpha_1'r)^3$$
$$-3(\alpha_1'\alpha_1)(\alpha_1'r_1)$$

Similarly , the large sample approximation for the estimation error for the estimator b_2^*, is

$$(b_2^* - \beta) = \emptyset_{-\frac{1}{2}} + \emptyset_{-1} + \emptyset_{-\frac{3}{2}} + \emptyset_{-2} + 0\left(T^{-\frac{3}{2}}\right)$$

Where

$$\emptyset_{-\frac{1}{2}} = (X'X)^{-1}X'u$$

$$\emptyset_{-1} = -\{g'(1)\sigma_2^2 + \left(1 - g'(1)\right)\sigma^2\}d\beta$$

$$\emptyset_{-\frac{3}{2}} = -\left(1 - g'(1)\right) d\beta - \{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}. d(X'X)^1X'u$$

$$\emptyset_{-2} = -\left(1 - g'(1)\right)d(X'X)^{-1}X'u + \frac{\left(1 - g'(1)\right)}{T(T-p)}(u'XX'u)d\beta$$
$$+ \{g'(1)\sigma_0^2 + (1 - g'(1)\sigma^2)\}^2. d^2\beta$$

From (3.3.10), we have

$$r_2 = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b_2^* - \beta)$$
$$= \zeta_0 + \zeta^*_{-\frac{1}{2}} + \zeta^*_{-1} + \zeta^*_{-\frac{3}{2}} + 0_p(T^{-2})$$

Where

$$\zeta_0 = Z$$

$$\zeta^*_{-\frac{1}{2}} = -\frac{\lambda}{\sigma}da_1$$

$$\zeta^*_{-1} = -\frac{\left(1 - g'(1)\right)}{\sigma} \in da_2 - \lambda dZ$$

$$\zeta^*_{-\frac{3}{2}} = -\left(1 - g'(1)\right) \in dZ + \frac{\sigma\left(1 - g'(1)\right)}{nT}(Z'X'XZ)da_2 + \frac{\lambda^2}{\sigma}d^2a_2$$

Proceeding on some line as for r_1, the characteristic function of vector r_2 to order $0_p\left(T^{-\frac{3}{2}}\right)$ is given by

$$\zeta_{r_2}(h) = \left(1 + \xi^*_{-\frac{1}{2}} + \xi^*_{-1} + \xi^*_{-\frac{3}{2}}\right)e^{-\frac{1}{2}h'h}$$

Where

$$\xi^*_{-\frac{1}{2}} = -\frac{i\lambda}{\sigma}d(h'\alpha_2)$$

$$\xi^*_{-1} = \lambda d(h'h) - \frac{\lambda^2}{2\sigma^2}(h'\alpha_2)^2$$

$$\xi^*_{-\frac{3}{2}} = \frac{i\left(1-g'(1)\right)}{nT}\sigma d(tr\,(X'X) - h'X'Xh)(h'\alpha_2) + \frac{i\lambda^2}{\sigma}d^2(h'\alpha_2) + \frac{i\lambda^2}{\sigma}d^2(h'\alpha'_2)h'h + \frac{i\lambda^3}{6\sigma^3}d^3(h'\alpha_2)^3$$

Using the approximate expression (13) of the characteristic function of r2, the inversion theorem (9) and the results in (8), we obtain the large sample approximate expression for the joint probability density function of r2, to order $0\left(T^{-\frac{3}{2}}\right)$, to be

$$f(r_2) = \left(1 + Y^*_{-\frac{1}{2}} + Y^*_{-1} + Y^*_{-\frac{3}{2}}\right)\xi(r_2)$$

Where

$$Y^*_{-\frac{1}{2}} = -\frac{d}{\sigma}\left(g'(1)\right)\sigma^2 + (1 - g'(1)\sigma^2)(\alpha'_2 r_2)$$

$$Y^*_{-1} = d\left(g'(1)\right)\sigma_0^2 + (1 - g'(1)\sigma^2)(p - r'_2 r_2) - \frac{d^2}{2\sigma^2}\left(g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\right)^2.(\alpha'_2\alpha_2 - (\alpha'_2 r_2)^2)$$

$$Y^*_{-\frac{3}{2}} = \frac{\left(1-g'(1)\right)d\sigma}{nT}\{(\alpha'_1 r_2)(r'_2 X'Xr_2) - 2(\alpha'_2 X'Xr_2)\} + \frac{d^2}{\sigma}\left(g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\right)^2(\alpha'_2 r_2)(r'_2 r_2) - (p+1)(\alpha'_2 r_2) - \frac{d^3}{6\sigma^3}\left(g'(1)\sigma_0^2 + \left(1-g'(1)\right)\sigma^2\right)^3.\{(\alpha'_2 r_2)^3 - 3(\alpha'_2\alpha_2)(\alpha'_2 r_2)\}.$$

For the derivation of the small disturbance asymptotic expressions of the sampling distributions of the estimators $b_1^*\ and\ b_2^*$, we rewrite the model taking $u = \sigma w$ as

$$Y = X\beta + \sigma w$$

So that w follows a multivariate normal distribution with mean vector zero and dispersion matrix I_T . The small disturbance approximation for the estimation error of estimator $b_1^*\ can$ be written from chapter II as

$$(b_1^* - \beta) = \sigma\phi_1 + \sigma^2\phi_2 + \sigma^3\phi_3 + \sigma^4\phi_4 + 0(\sigma^5)$$

Where

$$\phi_1 = (X'X)^{-1}X'w$$

$$\phi_2 = -V(X'X)^1 H\beta$$

$$\phi_3 = -V(X'X)^{-1}H(X'X)^{-1}X'W$$

$$\phi_4 = V^2(X'X)^{-1}H(X'X)^{-1}H\beta$$

$$V = g'(1)\mu + \frac{\left(1 - g'(1)\right)}{n}(W'\bar{P}_X W)$$

$$\mu = \frac{\sigma_0^2}{\sigma^2}$$

It may be mentioned here that the random vector $Z = (X'X)^{-\frac{1}{2}}X'W$ follows a multivariate normal distribution with mean vector zero and dispersion matrix I_T and $W\bar{P}_X W$ has a chi square distribution with n = (T-p) degrees of freedom. Also Z and $W\bar{P}_X W$ are independently distributed.

From, we have

$$r_1 = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b_1^* - \beta)$$

$$= A_0 + \sigma A_1 + \sigma^2 A_2 + \sigma^3 A_3 + \sigma^4 A_4 + 0(\sigma^4)$$

Where

$$A_0 = Z$$

$$A_1 = -V\alpha_1$$

$$A_2 = -VCZ$$

$$A_3 = V^2 C\alpha_1$$

The approximate characteristic function of the vector r_1 in up to order $0(\sigma^3)$ *is*

$$K_{r_1}(h) = E\left(e^{1h'r_1}\right)$$

$$= E\left\{e^{ihA_0}\, e^{\sigma ih'A_1 + \sigma^2 ih'A_2 + \sigma^3 ih'A_3}\right\}$$

$$= E\left[e^{ih'A_0\left\{1+\sigma(ih'A_1)+\sigma^2\left((ih'A_1)+\frac{1}{2}(ih'A_1)^2\right)+\sigma^3\left((ih'A_1)+(ih'A_1)(ih'A_1)+\frac{1}{6}(ih'A_1)^3\right)\right\}}\right]$$

Utilizing the results

$$E(W'\bar{P}_X W) = n$$

$$E(W'\bar{P}_X W)^2 = n(n+2)$$

$$E(W'\bar{P}_X W)^3 = n(n+2)(n+4),$$

And the results in(3.3.5) , we observe that the

Characteristic function of the vector r_1 up to order $0(\sigma^3)$, is given by

$$K_{r_1}(h) = (1 + \sigma K_1 + \sigma^2 K_2 + \sigma^3 K_3)e^{-\frac{1}{2}h'h}$$

Where

$$K_1 = -i\{\mu g'(1) + (1 - g'(1))\}(h'\alpha_1)$$

$$K_2 = \{\mu g'(1) + (1 - g'(1))\}(h'ch) - \frac{1}{2}\left\{\mu^2(g'(1))2 + (2\mu g'(1))(1 - g'(1)) + \frac{(n+2)}{n}(1 - g'(1))^2\right\}(h'\alpha_1)^2$$

$$K_3 = i\left\{\mu^2(g'(1))^2 + (2\mu g'(1))(1 - g'(1)) + \frac{(n+2)}{n}(1 - g'(1))^2\right\}(h'c\alpha_1) + i\left\{\mu^2(g'(1))^2 + (2\mu g'(1))(1 - g'(1)) + \frac{(n+2)}{n}(1 - g'(1))^2(h'\alpha_1)(h'ch)\right\} + \frac{i}{6}\left\{\mu^2(g'(1))^2 + 3\mu^3(g'(1))^2(1 - g'(1))\right\}$$

Now with the help of the result (8) and inversion theorem (3.3.8), the small disturbance asymptotic expression for the joint probability density function of the elements of vector r1 is given by

$$g(r_1) = (1 + \sigma v_1 + \sigma^2 v_2 + \sigma^3 v_3)\xi(r_1)$$

Where

$$v_1 = -\{g'(1)\mu + (1 - g'(1))\}(\alpha_1' r_1)$$

$$v_2 = \{\mu g'(1) + (1 - g'(1))\}(tr\, c - r' c\, r) - \frac{1}{2}\left\{\mu^2 (g'(1))^2 + 2\mu g'(1)(1 - g'(1)) + (1 - g'(1))^2 \frac{(n+2)}{n}\right\}(\alpha_1'\alpha_1 - (\alpha_1' r_1)^2)$$

$$v_3 = \left\{\mu^2 (g'(1))^2 + 2\mu g'(1)(1 - g'(1)) + (1 - g'(1))^2 \frac{(n+2)}{n}\right\} \ldots\ldots\ldots\ldots\ldots \{(\alpha_1' r_1)(r_1'\, c\, r_1) - (\alpha_1' r_1)(tr\, c - \alpha_1'\, c\, r_1)\}$$
$$- \frac{1}{6}\left\{\mu^2 (g'(1))^2 + 3\mu^2 (g'(1))^2 (1 - g'(1)) + 3\mu^2 (g'(1))^2 (1 - g'(1)) + 3\mu g'(1)(1 - g'(1)).\frac{(n+2)}{n} + (1 - g'(1))^3 . \frac{(n+2)(n+4)}{n^2}\right\}.\{(\alpha_1' r_1)^3 - 3(\alpha_1'\alpha_1)(\alpha_1' r_1)\}$$

From chapter II , we have small disturbance asymptotic expression for the estimation error of the estimator b_2^* to be

$$(b_2^* - \beta) = \sigma\emptyset_1 + \sigma^2\emptyset_2^* + \sigma^3\emptyset_3^* + \sigma^4\emptyset_4^* + 0(\sigma^5)$$

Where

$$\emptyset_1 = (X'X)^{-1}X'w$$

$$\emptyset_2^* = -vd\beta$$

$$\emptyset_3^* = -vd(X'X)^{-1}X'w$$

$$\emptyset_4^* = v^2 d^2 \beta$$

From (3.3.20), we have

$$r_2 = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b_2^* - \beta)$$

$$= A_0 + \sigma A_1^* + \sigma^2 A_2^* + \sigma^3 A_3^* + 0(\sigma^4)$$

Where

$$A_0 = Z$$

$$A_1^* = -vd\alpha_2$$

$$A_2^* = -vdz$$

$$A_3^* = v^2 d^2 \alpha_2$$

The approximate characteristic function of the vector r_2 in (3.3.21) upto order $0(\sigma^3)$ is

$$K_{r_2}(h) = [1 + \sigma K_1^* + \sigma^2 K_2^* + \sigma^3 K_3^*]e^{-\frac{1}{2}h'h}$$

Where

$$K_1^* = -i\{g'(1) + (1 - g'(1))\}(h'\alpha_2)$$

$$K_2^* = d\{\mu g'(1) + (1 - g'(1))\}h'h - \frac{d^2}{2}\left\{\mu^2(g'(1))^2 + 2\mu g'(1)(1 - g'(1))\right\} + \left\{\frac{(n+2)}{n}(1 - g'(1)^2)\right\}(h'\alpha_2)^2$$

$$K_3^* = id^2\left\{\mu^2(g'(1))^2 + 2\mu g'(1)(1 - g'(1)) + \frac{(n+2)}{n}(1 - g'(1))^2\right\}(h'\alpha_2)$$
$$- id^2\left\{\mu(g'(1))^2 + 2\mu(g'(1))(1 - g'(1)) + \left(\frac{n+2}{n}\right)(1 - g'(1))^2\right\}.(h'\alpha_2)(h'h)$$
$$+ \frac{i}{6}d^3\left\{\mu^3(g'(1)) + 3\mu^2(g'(1))^2(1 - g'(1)) + 3\left(\frac{n+2}{n}\right)\mu g'(1).(1 - g'(1))^2 + \left(\frac{n+2}{n}\right)\left(\frac{n+4}{n}\right)(1 - g'(1))^3\right\}(h'\alpha_2)^3$$

Using the approximate expression (3.3.22) of the characteristic function of r_2, the inversion theorem and the results in the small distribution asymptotic expression for the joint probability density function of the elements of vector r_2 is given by

$$g(r_2) = (1 + \sigma\, v_1^* + \sigma^2\, v_2^* + \sigma^3\, v_3^*)\xi(r_2)$$

Where

$$v_1^* = -d\{\mu g'(1) + (1 - g'(1))\}(\alpha_2'\, r_2)$$

$$v_2^* = d\left(\mu g'(1) + (1 - g'(1))\right)(p - r_2' r_2) - \frac{d^2}{2}\left\{\mu^2\left(g'(1)\right)^2 + 2\mu g'(1)\left(1 - g'(1)\right) + \left(1 - g'(1)\right)^2 \frac{n+2}{2}(\alpha_2'\alpha_2 - (\alpha_2' r_2)^2)\right\}$$

$$v_3^* = d^2\left\{\mu^2\left(g'(1)\right)^2 + 2\mu g'(1)\left(1 - g'(1)\right)\right\} + \left\{\left(1 - g'(1)\right)^2 \frac{(n+2)}{2}\right\} \ldots\ldots\ldots.\{(\alpha_2' r_2)(r_2' r) - (p+1)(\alpha_2' r_2)\} - \frac{d^3}{6}\left\{\mu^3\left(g'(1)\right)^3 + 3\mu^2\left(g'(1)\right)^2\left(1 - g'(1)\right) + 3\mu g'(1)\left(1 - g'(1)\right)^2 . \frac{(n+2)}{n} + \left(1 - g'(1)\right)^3 \frac{(n+2)(n+4)}{n^2}\right\}.\{(\alpha_2' r_2)^3 - 3(\alpha_2'\alpha_2)(\alpha_2' r_2)\}$$

Before finding the concentration probability of the generalized quasi minimax and mock-minimax estimators $b_1^*\ and\ b_2^*$ we find the concentration probability of the ordinary least square (OLS) estimators b around the true unknown coefficient vector β.

For p arbitrarily chosen positive constants $\bar{m}_1, \bar{m}_2, \bar{m}_3, \ldots\ldots\ldots.\bar{m}_p\ and\ \bar{m} = \left(\bar{m}_1, \bar{m}_2\,, \bar{m}_3, \ldots\ldots\ldots.\bar{m}_p\right)$, the concentration probability of of the OLS estimator b around β in the region bounded by the plane $\{|b - \beta| \le \bar{m}\} = \left\{\left|b_j - \beta\right| \le \bar{m}_j\ ;\ j = 1,2,3, \ldots\ldots p\right\}$in the p-dimensional ecludian space is given by

$$CP(b) = P\{|b-\beta| \leq \bar{m}\}$$

$$= P\left\{\frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b-\beta) \leq m\right\}, where\ m = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}\,\bar{m}$$

$$= P\{|Z_j| \leq m_j\ ; j = 1,2,3, \ldots \ldots .. p\}$$

$$= \emptyset(m),$$

Where m_j is the j^{th} element of the px1 vector m,Z_j's

(j = 1,2,3,………p) are the p elements of the standard normal vector $z = \frac{1}{\sigma}(X'X)^{\frac{1}{2}}(b-\beta)\ and\ \emptyset(m)$ is the probability that the standard normal Z lies in a region bounded by the planes $\left(|Z_j| \leq m_j\ ; j = 1,2,3 \ldots \ldots ., p\right)$ in the p-dimensional Ecludian space and is obtained by

$$\emptyset(m) = \int_{-m_p}^{m_p} \ldots \ldots \ldots \ldots \int_{-m_1}^{m_1} \xi(z)\ dz_1 \ldots \ldots \ldots \ldots \ldots \ldots .. dz_p$$

with $\xi(z) = \frac{1}{(2\pi)^{\frac{p}{2}}}\ e^{-\frac{1}{2}Z'Z}$ being the standard multivariate normal density of Z.

Proceeding on the same lines as for the OLS estimator b for finding the concentration probability, we see that the concentration probabilities of the estimators $b_1^*\ and\ b_2^*$ around β are given by

(1)Under large sample asymptotic

$$CP(b_1^*) = P\{|b_1^* - \beta| \leq \bar{m}\}$$

$$= P\{|r_{1j}| \leq m_j\ ; j = 1,2,3 \ldots \ldots \ldots p,\}$$

(where r_{1j} is the j^{th} element of vector r_1)

$$= \int_{-m_p}^{m_p} \ldots \ldots \ldots \ldots \ldots . \int_{-m_1}^{m_1} f(r_1)\ dr_{11}\ dr_{12} \ldots \ldots .. dr_{1p}$$

$$CP(b_2^*) = P\{|b_2^* - \beta| \leq \bar{m}\}$$

$$= P\{|r_{2j}| \leq m_j\ ; j = 1,2,3, \ldots \ldots . p\},$$

$$= \int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} f(r_2)\ dr_{21}\ dr_{22} \dots\dots..dr_{2p}$$

And

(2)Under small sigma asymptotic

$$CP(b_1^*) = \int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} g(r_1)\ dr_{11}\ dr_{12} \dots\dots..dr_{1p}$$

$$CP(b_2^*) = \int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} g(r_2)\ dr_{21}\ dr_{22} \dots\dots..dr_{2p}$$

Using the following result in

$$\int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} \xi(r)\ dr_1\ dr_2 \dots\dots..dr_p = \emptyset(m)$$

$$\int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} (\alpha' r)\, \xi(r)\ dr_1\ dr_2 \dots\dots..dr_p = 0$$

$$\int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} (r' A r)\, \xi(r)\ dr_1\ dr_2 \dots\dots..dr_p$$

$$= \left\{ tr\ A - \Sigma_{i=1}^{p} a_{ii} \frac{m_i e^{-\frac{1}{2}m_1^2}}{\int_0^{m_i} e^{-\frac{1}{2}r_i^2}\, dr_i} \right\} \emptyset(m)$$

$$\int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} (\alpha' r)^2\, \xi(r)\ dr_1\ dr_2 \dots\dots..dr_p$$

$$= \left\{ \alpha'\alpha - \Sigma_{i=1}^{p} a_i^2 \frac{m_i e^{-\frac{1}{2}m_1^2}}{\int_0^{m_i} e^{-\frac{1}{2}r_i^2}\, dr_i} \right\} \emptyset(m)$$

$$\int_{-m_p}^{m_p} \dots\dots\dots\dots\dots \int_{-m_1}^{m_1} (\alpha' r)(r' A\, r)\, \xi(r)\ dr_1\ dr_2 \dots\dots..dr_p = 0$$

Where a_{ii} is the i^{th} diagonal element of matrix A and α_i

is the i^{th} element vector α, we get the concentration probabilities of estimators b_1^* *and* b_2^* around β in the region bounded by the column vector m in the p-dimensional Ecludian space as

(1)Under large sample asymptotic to order $0\left(T^{-\frac{3}{2}}\right)$

$$CP(b_1^*) = \Big[1 + \{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}tr\, C\, E - \frac{1}{2\sigma^2}\{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}^2(\alpha_1'\, E\, \alpha_1)\Big].\emptyset(m)$$

$$CP(b_2^*) = \Big[1 + d\{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}tr\, E - \frac{d^2}{2\sigma^2}\{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}^2(\alpha_2'\, E\, \alpha_2)\Big].\emptyset(m)$$

where E is a diagonal matrix, i.e. E =diag. ($e_1, e_2, \ldots e_p$) with elements as

$$e_j = \frac{mj\, e^{-\frac{1}{2}m_j^2}}{\int_0^{m_j} e^{-\frac{1}{2}r_{1j}^2}\, dr_{ij}}; j = 1,2,3, \ldots \ldots p,$$

And

(ii) Under small sigma asymptotic to order $0(\sigma^3)$

$$CP(b_1^*) = \Big[1 + \{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}tr\, C\, E - \frac{1}{2\sigma^2}\{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}^2 + \frac{2}{n}\big(1 - g'(1)\big)^2\sigma^4(\alpha_1'\, E\, \alpha_1)\Big].\emptyset(m)$$

$$CP(b_2^*) = \Big[1 + d\{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}tr\, E - \frac{d^2}{2\sigma^2}\{g'(1)\sigma_0^2 + \big(1 - g'(1)\big)\sigma^2\}^2 + \frac{2}{n} + \frac{2}{n}\big(1 - g'(1)\big)^2\sigma^4\, (\alpha_2'\, E\, \alpha_2)\Big].\emptyset(m)$$

4. SOME REMARKS

(a) The results obtained by shukla (1993) concerning the concentration probabilities of the estimators b_1 and b_2 may be easily seen to be the special cases of those obtained in this chapter for the generalized estimators b_1^* *and* b_2^*.

(b) From (3.3.24) and (3.3.23) upto the order $0\left(T^{-\frac{3}{2}}\right)$ we see that

$$CP(b_1^*) - CP(b_2^*) = \left[d\left\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right\} tr\, C\, E - \frac{1}{2\sigma^2}\left\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right\}^2 (\alpha_2'\, E\, \alpha_2)\right].\emptyset(m)$$

which is positive if

$$\beta' H (X'X)^{-\frac{1}{2}} E (X'X)^{-\frac{1}{2}} H\beta \leq \frac{2\sigma}{\left(g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right)}.tr\,(X'X)^{-\frac{1}{2}} E (X'X)^{-\frac{1}{2}} H,$$

Which hold for a sufficient condition

$$\overline{Ch}.\left[(X'X)^{-\frac{1}{2}} E (X'X)^{-\frac{1}{2}} H\right]\beta < \frac{2\sigma^2}{\left(g'(1)\,\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right)}.tr(X'X)^{-\frac{1}{2}} E (X'X)^{-\frac{1}{2}} H,$$

Which hold if

$$\frac{g'(1)}{1 + g'(1)} < \frac{\sigma^2}{\sigma_0^2}$$

Further , from (3.3.25) and (3.3.23) to order $0\left(T^{-\frac{3}{2}}\right)$, we have

$$CP(b_1^*) - CP(b) = \left[d\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}tr\,E - \frac{d^2}{2\sigma^2}\{g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\}^2(\alpha_2'\,E\,\alpha_2)\right].\emptyset(m)$$

Which is positive if and only if

$$\beta'(X'X)^{-\frac{1}{2}}E(X'X)^{-\frac{1}{2}}\beta < \frac{2\sigma^2}{\left(g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right)}.\left(\frac{\overline{Ch}\,.(AH^{-1})}{tr\;\;A\,(X'X)^{-1}}\right)tr\,E$$

And holds true at least as long as

$$\overline{Ch}.\left[H^{-1}(X'X)^{-\frac{1}{2}}E(X'X)^{-\frac{1}{2}}\right] < \frac{2\sigma^2}{\left(g'(1)\;\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right)}.\left(\frac{\overline{Ch}\,.(AH^{-1})}{tr\;\;A\,(X'X)^{-1}}\right)tr\,E$$

Under small disturbance approach, comparison of concentration probabilities of the estimators $b_1^*, b_2^*, b_1, b_2\ and\ b$ may be done on similar lines.

(c) In particular, for the estimators

$$b_{10}^* = [X'X + S^2(1 - (1 - u)^k)H]^{-1}X'Y$$

REFERENCES

UDGE, G. G., GrIFFITHS W. E., HILLS R. C., LUTKEPOHL H., and LEE T. C. (1985) The Theory and Practices of Econometrics, Second Edition. New York, Wiley. KADANE, J. B. (1971) Comparison of kClass Estimators when the Disturbances are Small, Econometrica, 39, 723-738. NAGAR, A. L. and N. C. KoKWaNI (1964) The Bias and Moment Matrix of a Mixed Regression Estimator, Econometrica, 32, 174-182. SHUKLA, P. (1993) Estimation of Constrained Parameters in Regression Models, Thesis submitted and accepted, Lucknow University, Lucknow, India. Allen, D. M. [1971], "Mean square error of prediction as a criterion for selecting variables," Technimetrics, 13, 469–475. Allen, D. M. [1974], "The relationship between variable selection and data augmentation and a method for prediction," Technimetrics, 16, 125–127. Andrews, D. F. [1971], "Significance tests based on residuals," Biometrical, 58, 139–148. Andrews, D. F. [1974], "A robust method for multiple linear regression," Technimetrics, 16, 523–531. Andrews, D. F. [1979], "The robustness of residual displays," in R. L. Launer and G. N. Wilkinson (Eds.), Robustness in Statistics, Academic Press, New York, pp. 19–32. Andrews, D. F., P. J. Bickel, F. R. Hampel, P. J. Huber, W. H. Rogers, and J. W. Tukey [1972], Robust Estimates of Location, Princeton University Press, Princeton, N.J. Anscombe, F. J. [1961]