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A Review of Roman Domination in Digraphs

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Abstract: Roman domination is a widely studied area in graph theory. The same concept has been studied in the case of directed graphs. In this paper we review research done in the area of Roman domination in digraphs. All important results are discussed in this paper.

Index Terms **– Roman Domination, Directed Graphs, Roman Domination Number.**

I. INTRODUCTION

Roman domination models the problem of defending the strategically important places of the Roman empire from the enemies. During his reign when his strength was diminished, he was not able to defend all strategically important places of his empire directly by deploying forces at every place. Instead, he adopted a simplifying strategy by deploying either one or two units of force at some places and leaving some places without force. If there is no force at a place to guard, the forces from an adjacent place (two places are adjacent if they are connected by road and the forces at one place can move effectively between the two places to defend against an attack) can come to defend the attack provided that there are two units of forces at that place. If there are two units at one place, then one of the units will protect the same place and the other unit can defend against an attack at a neighboring place. The problem then becomes one to find the minimum number of units in total that are required to defend all places in the empire.

By representing the strategically important places in the empire by nodes and the connectivity between two places by a direct edge connection, we can obtain a graph that models the problem. Let $G = (V, E)$ be a graph. We can define a function $f: V \to \{0,1,2\}$ such that every vertex v with $f(v) = 0$ has a neighbor u with $f(u) = 2$. This function is called a roman dominating function (RDF) of the graph G. The weight of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The minimum weight of an RDF on G (denoted by $\gamma_R(G)$) is the Roman domination number of a digraph G. A $\gamma_R(G)$- function (or $\gamma_R$function) is a Roman dominating function of $G$with weight $\gamma_R(G)$. The concept of Roman domination was first proposed by Dreyer [1] in 2004.

This concept was extended to directed graphs by Kamaraj and Jakkammal [2] extended the idea of Roman domination to Digraphs, which was further studied by Sheikholeslami et al. [3]. Let $D = (V, A)$ be a directed graph. We define a function $f: V \to \{0,1,2\}$ such that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. This function is called a roman dominating function (RDF) of the graph D. The weight of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The minimum weight of an RDF on D(denoted by $\gamma_R(D)$) is the Roman domination number of a digraph D. A $\gamma_R(D)$- function (or $\gamma_R$function) is a Roman dominating function of $G$with weight $\gamma_R(D)$.

Let $u, v \in V(D)$. an arc from the vertex u to the vertex v denoted by (u, v) to denote the arc with direction from u to v. we say that v an out-neighbor of u u and an in-neighbor of v. The out-neighborhood of the vertex $v \in V(D)$ is denoted by $N^+(v)$ and it contains all vertices in $V(D)$such that there is an arc ending at that vertex and starting point at v. The in-neighborhood of v is the set of vertices in $V(D)$, having arcs directed from the vertex to the vertex v in D. It is denoted by $N(v)$.The closed outneighborhood is the set $N^+[v] = N^+(v) \cup \{v\}$. Similarly, the closed in -neighborhood of v is the set $N^-[v] = N^-(v) \cup \{v\}$. We define the out-degree of a vertex $v \in V(D)$ as $d^+(v) = d^+D(v) = |N^+(v)|$ and the in-degree of the vertex v as $d^-(v) = d^-D(v) = N^-(v)|$. The maximum and the minimum values of the out-degree, and in-degree are respectively, $\Delta^+(D)$, $\delta^+(D), \Delta^-(D), \delta^-(D)$.

II. SOME IMPORTANT RESULTS ON ROMAN DOMINATION IN DIGRAPHS

In this section, we review some important results on Roman domination in digraphs. Some of these results are taken from the Ph.D. thesis of Satheesh [5] and the related paper [7].

A Roman dominating function $f: V \to \{0,1,2\}$1can be represented by the ordered partition 2 (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$. Using this representation, weight if the RDF is $\omega(f) = V_1| + 2|V_2||$. Since $V_1 \cup V_2$is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, we have.

$$\gamma(D) \leq \gamma_R(D) \leq 2\gamma(D) \qquad (1)$$

Here γ (D) is the domination number of a digraph. The following result is given by Lee [2]

Proposition 1 [2]

Let D be a digraph with order n and minimum in-degree $\delta^-(D) \geq 1$. Then $\gamma(D) \leq \frac{2n}{3}$.

Kamaraj et al. [2] proved the following results which are the extensions of the properties of Roman dominating functions in ordinary graphs.

Proposition 2 [2]

Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(D)$ the function of a digraph D. Then

$_+(D[V\ ]) \leq 1$ 1. Δ 1 .
$w \in V$ N^- 2. If ·, then $_D(w \cap V = \emptyset)_2$.

3. If $u \in V_0$, then $N^+(u) \cap V_1 \leq 2$. $N_D^+(u) \cap V_1 \leq 2$.
4. V_2 is a $\gamma(D)$ set of the induced sub-digraph $D[V_0 \cup V_2]$.
5. Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ D with $N_-(v) \cap V_2 \neq \emptyset$ has at least two private neighbors relative to V_2 in the sub-digraph H.

They also gave an upper bound for Roman domination number of digraphs. **Proposition 3**

[2]

Let D be a d graph with order n. Then

$$\gamma_R(D) \leq n - \Delta^+(D) + 1 \qquad (2)$$

According to the characterization given by Sheikholeslami and Volkmann [3], the domination number and the Roman domination number of a digraph are equal if and only if $\Delta^+(D) = 0$. They proved the existence of a lower bound in terms of the order or domination number of the same graph.

Proposition 4 [3]

If D is a digraph on n vertices, then

$$\gamma_R(D) \geq min\{n, \gamma(D) + 1\} \quad (3)$$

Further, the characterized the digraphs such that their Roman domination number equals $\gamma(D) + 1$ and $\gamma(D) + 2$. Another result gives the condition for the Roman domination number to be less than the order of the graph. It indicates that many graphs are having equal Roman domination numbers and order.

Proposition 5 [3]

Let D be a digraph of order n. Then $\gamma_R(D) < n$ if and only if $\Delta^+(D) \geq 2$.

The above result has an immediate corollary.

Corollary 6 [3]

If D *is* a directed path or directed cycle of order n, then $\gamma_R(D) = n$.

Next, we discuss some techniques, which would simplify the task of determining the Roman domination number of digraphs. It is observed that in a directed cycle the number of vertices of in-degree and out-degree is the same. Using this fact the following fact has been proved in [5]. **Proposition 7 [5]**

If a directed cycle C of order n has r vertices of in degree 2, then $\gamma_R(C) \leq n - r$.

The above result is used in [5] to improve the upper bound for digraphs which contains a Hamiltonian cycle in the underlying graph. Suppose the orientation of the edges on the Hamiltonian circuit contains r vertices of in degree 2. Using the above result we get the following.

Proposition 8 [5]

If a directed graph D of order n contains a Hamilton a cycle C, which has r vertices of in-degree 2 together which has orientation, then $\gamma_R(C) \leq n - r$.

The following result is proved in [5] which express the Roman domination number of a digraph in terms of the Roman domination number of the induced sub-graphs $\langle V_1 \rangle$ and $\langle V_2 \rangle$ of the partition V_1 and V_2.

Proposition 9 [5]

Let V_1 and V_2 be a partition of $V(D)$ into two subsets, such that D doesn't contain an arc from $\langle V_1 \rangle$ to $\langle V_2 \rangle$. Then $\gamma_R(D) \leq \gamma_R(\langle V_1 \rangle) + \gamma_R(\langle V_2 \rangle)$.

Sheikholeslami et al [6] characterized the digraphs which attain the lower bound in (1).

Proposition 10 [6]

Let D be a digraph on n vertices. Then $\gamma(D) = \gamma_R(D)$ if and only if $\Delta^{+}(D) = 0$.

The following two characterizations proved by Sheikholeslami et al [6] are also notable.

Proposition 11 [6]

Let D be a digraph on $n \geq 2$ vertices with $\delta^{-}(D) \geq 1$. Then $\gamma_R(D) = \gamma(D) + 1$ if and only if there is a vertex $v \in V(D)$ with $d^{+}(v) = n - \gamma(D)$.

Proposition 12 [6]

Let D be a digraph on $n \geq 7$ vertices with $\delta^{-}(D) \geq 1$. Then $\gamma_R(D) = \gamma(D) + 2$ if and only if

(i) D does not have a vertex of outdegree $n - \gamma(D)$.

(ii) either D has a vertex of out-degree $n - \gamma(D) - 1$ or D contains two vertices v, w such that $|N^{+}[v] \cup [N^{+}[w]| = n - \gamma(D) - 2$.

III. CONCLUSION

In this short review on Roman domination in digraphs we discuss the developments in this research area and present the most important results. Very interesting results were proved in [6] and these results are either generalized or extended to a wider class of graphs in. This area of research is still active with any open problems. The exact value of Roman domination number is still unknown for many classes of graphs. The research in this area can be widened considering this aspects.

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