IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

THE DIOPHANTINE EQUATIONS $(x^2-c)^2 = (t^2 \pm 1)y^2 + l$

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ABSTRACT

$If\ c \neq |1|.\quad Then\ paper\ proposes\ to\ the\ equation\ (x^2-c)^2(t^2 \pm 1)y^2 + l\ have\ finitely\ many\ sloutions\ is\ positive\ integers\ x, y, t.$

Keywords: $Diophantine\ equations, Rational\ integer.$

Introduction

It is the object of this note to demonstrate that the two equations of the title have only finitely many solutions in positive integers x, y, t for any given integer $c \neq |1|$ and to provide a method for finding all the solutions by reducing the problem to finitely many Diophantine equations in two variables, each of which will have only finitely many solutions in integers. The cases $c \neq |1|$ are in principle similar, except that there may be rather trivial infinite families of solutions.[1]

The results are somewhat exceptional in that for every fixed $k \neq |2k_1^2|$, there are infinitely many values of c for which the equation $(x^2-c)^2 = (t^2+k)y^2 + 1$ has infinitely many solutions in positive integers x, y, t.

In the first place any solutions with $x^2 - c \leq 0$ and/or $t^2 - 2 < 0$ are finite in number and can be found by simple enumeration. Secondly if $t^2 - 2 = 2, i.e. t = 2$, we find since $x^2 - c > 0$ that

$$(x^2-c)^2-2y^2=1,$$

$$(x^2-c)+\left(y\sqrt{2}\right)^{2n}, n\geq 1.$$

Thus

$$x^2-c=\left\{\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1-\sqrt{2}\right)^{2n}}{2}\right\},$$

$$x^2-c(-1)^{n-1}=\left\{\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1-\sqrt{2}\right)^{2n}}{\sqrt{2}}\right\}^2$$

$$=z^2, \text{say},$$

where z is a rational integer. Thus $x^2-z^2=c\pm 1$, which can be solved immediately, giving only finitely many possible values for $x, if\ c\neq |1|$, hence only finitely many possible corresponding values for y. We therefore suppose from now on that $x^2-c>0$ and that $D=t^2\pm 2\geq 3$.

Consider first the case $D=t^2-2$, where the equation $u^2-Dy^2=2$ has solutions, with fundamental solution a, say. Then the fundamental solution of $u^2-Dy^2=1$ is $\beta=\frac{1}{2}a^2$. If now $(x^2-c)^2-Dy^2=1$, then

$$(x^2-c)+y\sqrt{D}=\beta^n=\left(\frac{1}{2}a^2\right)^{2n},$$

i.e. $$x^2-c=\frac{a^{2n}+a'^{2n}}{2^{n+1}}.$$

Then $x^2-c+1=\frac{\left(a^n+a'^n\right)^2}{2^{n+1}}$,

Since $aa'=2$. If n is odd, this yields $x^2-c+1=z^2$ where z is a rational integer, and this is easily solved. If $n=2m$ is even, then

$$x^2-c+1=2z^2,$$

Where

$$x^2=\frac{a^{2m}+a'^{2m}}{2^{m+1}}=-1+\frac{\left(a^m+a'^m\right)^2}{2^{m+1}}=y^2-1\ or\ 2y^2-1,$$

Where v is a rational integer, according as m is odd or even. Thus, we obtain either

$$x^2=2y^2-4y^2+(c+1)$$

Or $$x^2=8y^4-8y^2+(c+1),$$

and each of these equations has but a finite number of solutions in integers for each given $c \neq |1|$. Thus for each given c there are but finitely many possible values of x, and hence of corresponding y and t.

The case $D = t^2 + 2$ is entirely similar, except that now a is the fundamental solution of $u^2 - Dv^2 = -2, aa' = -2$ and $\beta = \frac{1}{2}a^2$. Then

$$x^2 - c + y\sqrt{D} = \beta^n = \left(\frac{1}{2}a^2\right)^{2n},$$

$$x^2 - c = \frac{a^{2n} + a'^{2n}}{2^{n+1}},$$

$$x^2 - c + (-1)^n = \frac{(a^n + a'^n)^2}{2^{n+1}},$$

If n is odd then $x^2 - c + 1 = z^2$, etc., as before. If $n = 2m$ is even, then

$$x^2 - c + 1 = 2z^2,$$

Where $z = \frac{a^{2m} + a'^{2m}}{2^{m+1}} = (-1)^{m+1} + \frac{(a^m + a'^m)^2}{2^{m+1}} = y^2 + 1$ or $2y^2 - 1$.

Thus, we obtain in this case, either

$$x^2 = 2y^4 - 4y^2 + (c + 1)$$

or $$x^2 = 8y^4 + 8y^2 + (c + 1),$$

and the result follows as before. This concludes the proof of the main result of the paper.

However, if $k \neq |2k_1^2|$, then the equation

$$(x^2 - c)^2 = (t^2 + k)y^2 + 1$$

is satisfied by integers x, y, t where $y = 2u, t = |k|u$ provided

$$(x^2 - c)^2 = (k^2u^2 + k) \cdot 4u^2 + 1 = (2ku^2 + 1) + 1,$$

i.e., provided that either

$$x^2 - 2ku^2 = c + 1$$

or $$x^2 + 2ku^2 = c - 1,$$

and since $k \neq |2k_1^2|$, one of these equations has infinitely many solutions for suitable values of c.

Reference

[1] J. H. E. Cohn, The Diophantine equation $(x^2 - c)^2 = dy^2 + 4$, J. London Math. Soc. (2) 8 (1974), p. 253