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RESULT ON COMMON FIXED POINT IN FUZZY METRIC SPACE FOR FOUR COMPATIBLE MAPPINGS

V.H.Badshah and Prerna Sharma

School of Studies in Mathematics, Vikram University, Ujjain (M.P.)-456010,India

Abstract: The present paper set up a common fixed point theorem in a complete fuzzy metric space for four compatible mappings.

Keywords and Phrases: R-Weakly Commuting Pairs, Compatible Mapping, Self Mappings, Complete Fuzzy Metric Space , Common Fixed Point .

AMS (2010) Subject Classifications: Primary 54H25, Secondary 47H10.

1. INTRODUCTION

Zadeh [30] was a professor emeritus of mathematician with computer scientist, electrical engineer artificial intelligence researcher and best known for proposing fuzzy mathematics consisting of these fuzzy related concepts. Kramosil and Michalek [11] had put forwarded the contraction principal for fuzzy metric space . In general subrahmanyam [25,26] forwarded with generalized, established the result of Garabiec [6] for a pair of commuting mappings. R-Weakly commutativity of mappings in fuzzy metric spaces was defined by Vasuki [27]. In this progress lot of researchers namly namely George and Veeramani[4,5] ,Fuller [3],Gregori and Sapena [7],Imdad, Ali and Hasan [8],Mihet [14], Sastry,Naidu and Krishn [17],Schweizer [18],Bratney and Odeh [13],Romaguera ,Sapena and Tirado [16],Shirude and Aage [21], Steimann [24], Vijayaraju and Sajath [28], Singh and Jain [22] , Jungck [9], Amari and Moutawakil [1], Mujahid Abbas [2], Sedghi,et.al.[19], Khan [10],Pathak, Lopez and Verma [15], Shen,et.al.[20], Wairojjana, et.al. [29] Recently Soni and Shukla [23] established some fixed point theorem in fuzzy metric space for expansion mapping and proved under different conditions. Manthena and Manchala [12] proved two common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric space using property E.A.

2.PRELIMINARIES

For this purpose we need the following definitions and Lemmas.

Definition 2. 1. *The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:*

(2.1) $M(x, y, t) > 0$,

(2.2) $M(x, y, t) = 1$ *if and only if* $x = y$,

(2.3) $M(x, y, t) = M(y, x, t)$,

(2.4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + S)$,

(2.5) $M(x, y, .): (0, \infty) \rightarrow [0,1]$ *is continuous, for all* $x, y, z \in$ X and t, s > 0.

Definition 2.2. A Sequence $\{x_n\}$ in a fuzzy metric space $(X, M,*)$ is a Cauchy sequence if and only if for each Ɛ > 0, t > 0, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all n, m ≥ n_0.

Definition 2.3. Fuzzy metric space $(X, M,*)$ is said to be complete if every Cauchy sequence in $(X, M,*)$ is a convergent sequence.

A sequence $\{x_n\}$ in $(X, M,*)$ is convergent to x ∈ X if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for each } t > 0.$$

Definition 2.4. Two mappings f and g on a Fuzzy metric space $(X, M,*)$into itself are said to be Weakly Commuting if

$$M(fgx, gfx, t) \geq M(fx, gx, t) \text{ for each } x \text{ in } X.$$

Definition 2.5. The mapping f and g of a Fuzzy metric space $(X, M,*)$ into itself are said to be R-weakly commuting, provided there exists some positive real numbers R such that

$$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right) \text{ for each } x \text{ in } X.$$

Definition 2. 6. Self mappings F and G of a Fuzzy metric space $(X, M,*)$ are said to be compatible if and only if $M(FGx_n, GFx_n, t) \rightarrow 1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $Fx_n, Gx_n \rightarrow y$, for some y in X.

Definition 2. 7. Let A and S be self mappings of a Fuzzy metric space $(X, M,*)$. We will call A and S to be reciprocally continuous if
$\lim_{n\to\infty} ASx_n = Ap$ and $\lim_{n\to\infty} SAx_n = Sp$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

If A and S are continuous then they are obviously reciprocally continuous. But the converse need not be true.

Lemma 2.1. M(x, y, *) is non- decreasing for all x, y in X.

Lemma 2.2. Let $\{y_n\}$ be a sequence in a Fuzzy metric space $(X, M, *)$. If there exists a number k ∈ (0,1) such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)\ \forall\ t > 0, n \in N$, then $\{x_n\}$ is Cauchy sequence in X.

3 MAIN RESULTS

Theorem 3.1. Let A_1, A_2, A_3 and A_4 be self maps of a complete Fuzzy metric space (X, M, *) with continuous t-norm * defined by

$a * b = \min\{a, b\}, a, b \in [0,1]$ satisfying

(a) $A_1(x) \subset A_4(x)$, $A_2(x) \subset A_3(x)$,

(b) [A_1, A_3] or [A_2, A_4] is compatible pair of reciprocally continuous maps,

(c) [A_1, A_3] ,[A_2, A_4] are point wise R-weakly commuting pairs of maps,

(d) For all x, y ∈ X, $l \in (0,1)$, t > 0

$$M^2(A_1x, A_2y, lt) \geq \emptyset\{min\{M^2(A_3x, A_4y, t), M^2(A_1x, A_3y, t), [M^2(A_2y, A_4y, t) + M^2(A_1x, A_4y, t)], [M(A_1x, A_4y, 2t) + M(A_2y, A_3x, 2t)]\}\}$$

where,

$\emptyset(t) : R^+ \rightarrow R^+$ is non-decreasing continuous from the right with $\emptyset(t) \geq t\ for\ t > 0\ and\ \emptyset(1) = 1$, $\lim_{t\to\infty} M(x, y, t) \rightarrow 1$.

Then $A_1, A_2, A_3\ and\ A_4$ have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrarily point in X, construct a sequence $\{z_n\}$ in X, such that

$$z_{2n-1} = A_4x_{2n-1} = A_1x_{2n-2}\ and\ z_{2n} = A_3x_{2n} = A_2x_{2n-1} \quad n = 1,2,3 \ldots \ldots$$

Now,

$$M^2(z_{2n+1}, z_{2n+2}, lt) = M^2(A_1x_{2n}, A_2x_{2n+1}, lt)$$

$$\geq \emptyset\left\{Min\{M^2(A_3x_{2n}, A_4x_{2n+1}, lt), M^2(A_1x_{2n}, A_3x_{2n}, t), \left[M^2(A_2x_{2n+1}, A_4x_{2n+1}, t) + M^2(A_1x_{2n}, A_4x_{2n+1}, t), [M(A_1x_{2n}, A_4x_{2n+1}, 2t) + M(A_2x_{2n+1}, A_3x_{2n}, 2t)]\right]\}\right\}$$

$$\geq \emptyset\{Min\{M^2(z_{2n}, z_{2n+1}, t), M^2(z_{2n}, z_{2n+1}, t), [M^2(z_{2n+2}, z_{2n+1}, t) + M^2(z_{2n+1}, z_{2n+1}, t)], [M(z_{2n+1}, z_{2n+1}, 2t) + M(z_{2n+2}, z_{2n}, 2t)]\}\}$$

$$\geq \emptyset\{Min\{M^2(z_{2n}, z_{2n+1}, t), [M^2(z_{2n+2}, z_{2n+1}, t) + 1], [1 + M(z_{2n}, z_{2n+2}, 2t)]\}\}$$

If $M(z_{2n+1}, z_{2n+2}, t) + 1 \leq M(z_{2n}, z_{2n+1}, t)$, then we get a contradiction, and so

$M^2(z_{2n+1}, z_{2n+2}, lt) \geq \emptyset[M^2(z_{2n}, z_{2n+1}, t)]$

$M(z_{2n+1}, z_{2n+2}, lt) \geq M(z_{2n}, z_{2n+1}, t), t > 0$ ………. (a)

Further using (d), we have

$M^2(z_{2n}, z_{2n+1}, lt) = M^2(A_2x_{2n-1}, A_1x_{2n}, lt)$

$= M^2(A_1x_{2n}, A_2x_{2n-1}, lt)$

$$\geq \emptyset\{Min\{M^2(A_3x_{2n}, A_4x_{2n-1}, t), M^2(A_1x_{2n}, A_3x_{2n}, t), [M^2(A_2x_{2n-1}, A_4x_{2n-1}, t) + M^2(A_1x_{2n}, A_4x_{2n-1}, t)], [M(A_1x_{2n}, A_4x_{2n-1}, 2t) + M(A_2x_{2n-1}, A_3x_{2n}, 2t)]\}\}$$

$$\geq \emptyset\{Min\{M^2(y_{2n}, y_{2n+1}, t), M^2(y_{2n}, y_{2n+1}, t), [M^2(y_{2n}, y_{2n+1}, t) + M^2(y_{2n+1}, y_{2n+1}, t)], [M(y_{2n+1}, y_{2n-1}, 2t) + M(y_{2n}, y_{2n}, 2t)]\}\}$$

$$\geq \emptyset\{Min\{M^2(y_{2n}, y_{2n+1}, t), M^2(y_{2n}, y_{2n+1}, t), [M^2(y_{2n}, y_{2n+1}, t) + 1], [M(y_{2n+1}, y_{2n-1}, 2t) + 1]\}\}$$

If $M(y_{2n}, y_{2n+1}, t) + 1 \leq M(y_{2n-1}, y_{2n}, t)$ then we get a contradiction and so

$M^2(y_{2n}, y_{2n+1}, lt) \geq \emptyset[M^2(y_{2n-1}, y_{2n}, t)]$ ……… (b)

$M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, t)$

Using (a) and (b), we have

$M(y_n, y_{n+1}, lt) \geq M(y_{n-1}, y_n, t) \;\forall\; t > 0$

This implies that $\{y_n\}$ is a Cauchy sequence by Lemma 2.2.

Since (X, M, *) is complete, so $\{y_n\}$ converges to some point z in X. Thus subsequences $\{A_1, x_{2n}\}$, $\{A_3, x_{2n}\}$, $\{A_{2,} x_{2n}\}$ and $\{A_4, x_{2n}\}$ also converges to z. Suppose $[A_1, A_3]$ is compatible pair of reciprocally continuous maps. Then by the definition of reciprocally continuous maps, $A_1, A_3x_{2n} \rightarrow A_1z$ and , $A_3, A_1x_{2n} \rightarrow A_3z$ and then the compatibility of A_1 and A_3 yields $\lim_{n\to\infty} M(A_1A_3x_{2n}, A_3A_1x_{2n}, t) = 1$

i.e. $M(A_1z, A_3z, t) = 1$. Hence, $A_1z = A_3z$

Since, $A_1(X) \subset A_4(X)$, there exists a point u in X such that $A_1z = A_4u$.

Using (iv) we have.

$M^2(A_1z, A_2u, lt) \geq \emptyset\{Min\{M^2(A_3z, A_4u, t), M^2(A_1z, A_3z, t), [M^2(A_2u, A_4u, t) + M^2(A_1z, A_4u, t)], [M(A_1z, A_4u, 2t) + M(A_2u, A_3z, 2t)]\}\}$

$$\geq \emptyset\{Min\{M^2(A_1z, A_1z, t), M^2(A_1z, A_1z, t), [M^2(A_2u, A_1z, t) + M^2(A_1z, A_1z, t)], [M(A_1z, A_1z, 2t) + M(A_2u, A_1z, 2t)]\}\}$$

$\geq \emptyset\{Min\{1, 1, M^2(A_2u, A_1z, t) + 1, 1 + M(A_2u, A_1z, 2t)\}\}$

Or $M^2(A_1z, A_2u, lt) \geq \emptyset[M^2(A_1z, A_2u, t)] \geq M^2(A_1z, A_2u, t)$ which implies that

$A_1z = A_2u$. Thus, $A_3z = A_1z = A_4u = A_2u$.

R-weakly commutativity of A_1 *and* A_3 implies that there exists $R > 0$ such that

$M(A_1A_3z, A_3A_1z, t) \geq M\left(A_1z, A_3z, \frac{t}{R}\right) = 1$

i.e. $A_1A_3z = A_3A_1z$

and $A_1A_1z = A_1A_3z = A_3A_1z = A_3A_3z$

Similarly R-weakly commutativity of A_2 *and* A_4 implies that

$A_2A_2u = A_2A_4u = A_4A_2u = A_4A_4u$

Now by (iv), we have

$M^2(A_1A_1z, A_1z, lt) = M^2(A_1A_1z, A_2u, lt)$

$M^2(A_1A_1z, A_2u, lt)$

$\geq \emptyset\{Min\{M^2(A_3A_1z, A_4u, t), M^2(A_1A_1z, A_3A_1z, t), [M^2(A_2u, A_4u, t) + M^2(A_1A_1z, A_4u, t)], [M(A_1A_1z, A_4u, 2t) + M(A_2u, A_3A_1z, 2t)]\}\}$

$\geq \emptyset\{Min\{M^2(A_1A_1z, A_1z, t), M^2(A_1A_1z, A_1A_1z, t), [M^2(A_1z, A_1z, t) + M^2(A_1A_1z, A_1z, t)], [M(A_1A_1z, A_1z, 2t) + M(A_1z, A_1A_1z, 2t)]\}\}$

Or $M^2(A_1A_1z, A_1z, lt) \geq \emptyset[M^2(A_1A_1z, A_1z, t)] \geq M^2(A_1A_1z, A_1z, t)$

$A_1A_1z = A_1z$

Thus, $A_1z = A_1A_1z = A_3A_1z$. Thus A_1z is a common fixed point of A_1 *and* A_3.

Again by (iv), we have

$M^2(A_1z, A_2A_2u, lt)$

$\geq \emptyset\{Min\{M^2(A_3z, A_4A_2u, t), M^2(A_1z, A_3z, t), [M^2(A_2A_2u, A_4A_2u, t) + M^2(A_1z, A_4A_2u, t)], [M(A_1z, A_4A_2u, 2t) + M(A_2A_2u, A_3z, 2t)]\}\}$

$\geq \emptyset\{Min\{M^2(A_1z, A_2A_2u, t), M^2(A_1z, A_1z, t), [M^2(A_2A_2u, A_2A_2u, t) + M^2(A_1z, A_2A_2u, t)], [M(A_1z, A_2A_2u, 2t) + M(A_2A_2u, A_1z, 2t)]\}\}$

Or $M^2(A_1z, A_2A_2u, lt) \geq \emptyset\{M^2(A_1z, A_2A_2u, t)\} \geq M^2(A_1z, A_2A_2u, t)$

This implies that

$A_1z = A_2A_2u$. Thus, $A_2A_2u = A_1z = A_4A_2u = A_2u$

Thus, A_2u $(= A_1z)$ is a common fixed point of A_2 and A_4 and hence A_1z is a common fixed point of A_1, A_2, A_3 and A_4.

To prove uniqueness, let A_1z_1 be another common fixed point of A_1, A_2, A_3 and A_4. Then

$M(A_1z, A_1z_1, lt) \geq M^2(A_1A_1z, A_2A_1z_1, lt)$

$$\geq \emptyset\{Min\{M^2(A_3A_1z, A_4A_1z_1, t), M^2(A_1A_1z, A_3A_1z, t), [M^2(A_2A_1z_1, A_4A_1z_1, t) + M^2(A_1A_1z, A_4A_1z_1, t)], [M(A_1A_1z, A_4A_1z_1, 2t) + M(A_2A_1z_1, A_3A_1z, 2t)]\}\}$$

$$\geq \emptyset\{Min\{M^2(A_1z, A_1z_1, t), M^2(A_1z, A_1z, t), [M^2(A_1z_1, A_1z_1, t) + M^2(A_1z, A_1z_1, t)], [M(A_1z, A_1z_1, 2t) + M(A_1z_1, A_1z, 2t)]\}\}$$

Or $M^2(A_1z, A_1z_1, lt) \geq \emptyset\{M^2(A_1z, A_1z_1, t)\} \geq M^2(A_1z, A_1z_1, t)$

and so $A_1z = A_1z_1$

Thus A_1z_1 is a unique common fixed point of A_1, A_2, A_3 and A_4.

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