IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

A Study of finsler spaces in Differential Geometry

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ABSTRACT

In a Finsler space with a Finsler metric L (X,Y), the fact that the fundamental tensor field g_{ij} (x,y) is provided as $g_{ij} = (\partial^2 L^2/\partial Y^i \partial y^j)/2$ from the fundamental function L plays an important role in the study of the Finsler space but is said to be not necessarily desirable from the point of view of theoretical physics.

INTRODUCTION

Finsler space is smooth Manifold possessing a finsler Metric. Fiinsler Geometry is R-Geometry without condition that the line element be quadratic. The purpose of the present paper is to study some properties of generalized Finsler space and then one or our metrical Finsler connections is used. CH-1 is devoted to the preliminaries. In CH-2 we shall consider generalized Finsler spaces corresponding to locally Minkowski spaces. In CH-3, we consider generalized Finsler spacer. Corresponding to Berwald spaces . And in CH-4 we deal with a special metrical Finsler strcture and, in a generalized Finsler space with this special metrical Finsler space with this special metrical Finsler structure, we discuss the results obtained in CH-2 and CH-3

The terminologies and notations in the present paper are referred to Matsumoto's book [4].

1. PRELIMINARIES

Let us quote the results necessary for the subsequent considerations from Matsumoto's book [4]. Let FΓ =(F_{jk}^i, N_j^i, C_{jK}^I) be a Finsler connection on an n –dimesional generalized Finsler space Mn. Then, with respect to the Finsler connection FΓ, the h-and v-covariant derivatives of a Finsler tensor field T_j^i, for example, are given by-

$$T_j^i|k = \delta_k T_j^i + T_j^h F_{hk}^i - T_h^i F_{jk}^h,$$
$$T_j^i|k = \delta_k T_j^i + T_j^h C_{hk}^i - T_h^i C_{jk}^h,$$

Where $\delta_k = \delta_k - N_j^i \delta_i, \delta_k = \delta/\delta_{x^k}$ and $\delta_k = \delta/\delta_{y^k}$. And, five torsion tensor fields and three curvature tensor fields of the Finsler connection FΓ are obtained as follows:

(1.1) C_{jk}^i ... (h) hv-torsion

(1.2) $R_{jk}^i = A_{jk}\{\delta_k N_j^i$ ... (v) h-torsion,

(1.3) $T_{jk}^i = A_{jk}\{F_{jk}^i\}$ ... (h) h-torsion,

(1.4) $P_{jk}^i = \delta_k N_j^i - F_{kj}^i$ ... (v) hv-torsion,

(1.5) $S_{jk}^i = A_{(jk)}\{C_{jk}^i\}$ ... (v) v-torsion,

(1.6) $R_{h\ ik}^i = A_{jk}\{\delta_k F_{hj}^i = F_{hj}^r F_{rk}^i\} + C_{hr}^i R_{jk}^r$ ... h-curvature,

(1.7) $P_{h\ jk}^i = \delta_k F_{hj}^i + C_{hkij}^i + C_{hr}^i P_{jk}^r$ ... h v-curvature,

(1.8) $S_{h\ jk}^i = A_{(jk)}\{\delta_k C_{hj}^i + C_{hr}^r C_{rk}^i\}$ ...v-curvature

Where $A_{(jk)}$ means the intercharge of the indices j, k in the terms enclosed with the parentheses { } and subtraction.

Hereafter, we shall deal with the Finsler connection $F_j^i = (F_{jk}^i, N_k^i, C_{jk}^i)$ for which the following conditions are satisfied:

(1.9) (a) $T_{jk}^i = 0,$ (b) $S_{jk}^i = 0,$

(1.10) $F_{jk}^i y^i = N_k^i,$

(1.11) (a) $g_{ij|k} = 0$ (b) $g_{ij|k} = 0$

A Finsler connection on Mn is called metricall if it satisfies the conditions (1.11) (a) and (b). In order to use later on, for the above metrical Finsler connection, let us introduce one of the Bianchi identities.

(1.12) $P_{ijk1} = P_{jik1}$

(1.13) $A_{(jk)}\{C_{khj|i} + C_{ihr} P_{ki}^r - P_{jhki}\} = 0$

Where $C_{khi} = g_{hr} C_{ki}^r$ and $P_{khi} = g_{hr} P_{ikl}^r$.

Now introduce an identity necessary for the subsequent consideration. Applying he Christoffel process [4] with respect to the indices j,k,h to the Bianchi identity (1.12) and using (1.13), we have-

(1.14) $2P_{jhki} = (C_{khi} + C_{hki}) + (C_{hji} - C_{jhi})_{jk} - (C_{kji} + C_{jki})_{jh} + (C_{jhi} - C_{hji})P_{ki}^r + (C_{kjr} + C_{jkr})P_{hi}^r - (C_{khr} + C_{hkr})P_{ji}^r$

Again we are going to determine a metrical Finsler connection $F\Gamma = (F^i_{jk}, N^i_j C^i_{jk})$ with the conditions (1.9) and (1.10).then, it is easily seen that—

(1.15) $C^i_{jk} = g^{ih}(\delta_j g_{hk} + \delta_k g_{hi} - \delta_h g_{jk})/2$

Let (A^i_j) be a matrix defined by—

(1.16) $A^i_j = \delta^i_j + g^{ir} y^s \delta_i g_{rs}$

Assuming that the matrix $(A^i_j)$is regular, we can consider a matrix (A^{pq}_{kj}), (pq, kj = 11,12, …, …, In, …, n1, n2 …, nn), defined by

(1.17) $A^{pq}_{kj} = 2\delta^q_k \delta^p_i + \delta^q_k g^{pr} y^i \delta_r g_{ij} + B^l_r C_{soo} g^{so} g^{qr} \delta_l g_{jk} - g^{rp} y^i \delta^q_j \delta_r g_{ki},$

Here the index o denotes the contraction by y and the matrix (B^l_r) is the inverse matrix of (A^i_j)

THEOREM 1:

Let Mn be a generalized Finsler space and let us assume that two matrices $(A^i_j)\ and\ (A^{pi}_{kj})$ are regular. Then, there exists uniquely a metrical Finsler connection $F\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ with the conditions (1.9) and (1.10).

It is given by (1.15), (1.10) and

(1.18) $F^i_{jk} = \gamma^i_{jk} - g^{ir}(F^s_{ok}\delta_s g_{kr} - F_{osr}\delta_s g_{jk})/2$, where

(1.19) $F_{osr} = g_{rj}\gamma^r_{jk} = (\delta_i g_{ik} + \delta_k g_{ji} - \delta_j g_{ik})/2$

(1.20) $\gamma_{ijk} = g_{rj}\gamma^r_{ik} = (\delta_i g_{ik} + \delta_k g_{ji} - \delta_j g_{ik})/2$

And the matrix (B^{kj}_{rs}) is the inverse matrix of (A^{pi}_{kj}).

In the subsequent considerations, we use the Finsler connection given by Theorem

I. Remarks1. In Finsler spaces, the metrical Finsler connection in theorem 1 reduces to the Cartan connection.

II. Exitence of a coordinate system (xi) in which gij are functions of ys only.

In this section, we shall prove the following theorem.

THEOREM 2

A necessary and sufficient condition that there exist a coordinate system (x^i) for a generalized Finsler space Mn for which the components of the metricalFinsler structure g_{ij} are function of y_i only is that

(2.1) $R_{ijk} = 0$

(2.2) $R_{hijk} = 0$

(2.3) $Ch_{ijlk} = 0$

(2.4) $P_{rjk} = 0$

Where $P_{ijk} = g_{ir}P^r_{jk}$

PROOF :

Assume that g_{ij} are functions of y^i only in a coordinate system (x^i). Then, using (1.18), (1.19) and (1.20), we having F_{jk}^i=0and then $N_k^i = F_{j|k}^i y^i = 0$. From this equation and (1.2), we obtain (2.1). Using (1.6), (2.2) results at once from (2.1) and the fact that $F_{jk}^i = 0$. Using (1.15), the assumption gives that C_{jk}^i are functions of y^i only. Accordingly, using $F_{jk}^i = 0$ and N_j^i=0, we have (2.3), (2.4) is obvious b (1.4).

Next, we shall prove that converse. From (2.3) and (2.4), (1.14) gives P_{jhki}=0. From this equation, (1.7), (2.3) and (2.4), we have $\delta_k F_{hj}^i = 0$, that is, F_{hj}^i are functions of position only. Accordingly, noticing (2.1) and (2.2), we have that (1.6) reduces to-

$$A_{(jk)}\{\delta_k F_{hj}^i + F_{hj}^r F_{rk}^i\} = 0.$$

From these facts, in the similar way to the case of a Riemannian space with zero curvature, we get that there exists a coordinate system ($\overline{X^a}$) such that $\overline{F_{bc}^a}$=0. Hence we have $\overline{N_b^a}$=0, so that (1.11) (a) gives that $\bar{g}^{ab}$ are functions' of y^{-a}only Q.E.D

Remark 2. For the Cartan connection in a Finsler space, (2.1) and (2.2) implies becaue of the identity $R_{hijk}y^h = R_{ijk}$ and (2.3) implies (2.4) because $C_{ijk|o} = P_{ijk}$. But in our case, the above stated facts are noe true, since we have that $R_{hlik}\ y^h = R_{ijk} + C_{oir}R_{jk}^r$ and tht $C_{ijk|o} = P_{ijk}$ is not necessarily satisfied.

2. Acondition for$F\ _{jk}$ to de functions of position only.
3. In this section we shall prove the following theorem.

THEOREM 3

A necessary and sufficient condition for the connection coefficients F_{jk}^i to be functions of position only in that the following equations are satisfied.

(3.1) P_{ijk}=0,

(3.2) $C_{khi\,|j} + C_{hki|j} = 0$

PROOF:

The fact that F^i_{jk} are functions of position only is equivalent to a conditions $\delta_h F^i_{jk} = 0$. And, from (1.7), the conditions

$\delta_h F^i_{jk} = 0$ is equivalent to –

(3.3) $p_{hijk} = -C_{hik|j} + C_{hir} p^i_{jk}$

Now we assume that $\delta_h F^i_{jk} = 0$. Then,

From (1.4), we have $p^i_{jk} = 0$. from this and (3.3), we have

(3.4)$p_{hjiki} = -C_{hjik}$

Noticing $p^i_{jk} = 0$, from (1.14) we have

(3.5)$2_{phjki} = c_{kji|h} + c_{jk|h} - c_{khi|j} - c_{hki|j-}\ c_{hki|j} - c_{hji|k} + c_{jhi|k}$

From (3.4) and (3.5),we have

(3.6) $c_{kii|h} + c_{jki|h} - c_{khi|j} - c_{hki|j} + c_{hji|k} + c_{jhi|k} = 0$

Cyclic permutations of indices h, j, k in (3.6)and summation yield

(3.7) $c_{kji|h} + c_{jki|h} + c_{khi|j} + c_{hki|j} + c_{hij|k} + c_{jki|k} = 0$

From (3.6)and (3.7), we have (3.2)

Next we shall prove the converse. Substituting (3.1) and (3.2) into (1.14), we have

$2^p_{jhki} = c_{hji|k} - c_{jhi|k} = -2_{cjhi|k,}$ thus we have (3.3)equivalent to the condition$\delta_{hF^i_{jk}}$=0.

Q.E.D

Remarks 3. For the cartan connection in a Finsler space, (3.2) implies (3.1).

4. A generalized Finsler space with
$g_{ij} = e^{2\sigma(x,y)} y_{ij}(x)$

In this section, we shall consider a generalized Finsler space Mn with a special metrical Finsler structure.

(4.1) $g_{ij} = e^{2\sigma(x,y)} y_{ij}(x)$

Where σ is positively homogeneous of degree 0 with respect to y and Yij is Ricmannian metric of Mn. Then, the following theorem is known.

THEOREM 4

[5], [6], Let Mn e a generalized Finsler space with gij given by (4.1). then, under the conditions (1.9) and (1.10), there exists uniquely a metrical Finsler connection

FΓ FΓ = $\left(F^i_{jk}, N^i_k, C^i_{jk}\right)$ on M^n.

From (1.15), we get easily

(4.2) $C_{ijk} = (C_i g_{uk} + C_k g_{ji} - C_{jgik})/n$

(4.3) $C_j = C_{iij} = n\delta_{i^\sigma}$

THEOREM 5

With respect to the metrical Finsler connection FΓ in theorem 4, the following three conditions are equivalen

(4.4) $C_{ijk|h} + C_{ijk|h} = 0$

(4.5) $C_{ijk|h} = 0$

(4.6) $C_{i|h} = 0$

PROOF :

Differentiating (4.2) h-covariantly with respect to xh we have

(4.7) $C_{ijk|h} = (C_{i|h} g_{jk} + C_{k|h} g_{ji} - C_{j|h} g_{ik})/n$

From this equation, we get $C_{ijk|h} + C_{k|h} = 2C_{k|h} g_{ij}$. Thus, (4.4) is equivalent to (4.6) Using (4.7), we have easily that (4.5) is equivalent to (4.6).

From theorem 5, in the case of the generalized Finsler space considered in this section, we have that the condition (2.3) (resp. (3.2) in Theorem 2 (resp. Theorem3) is equivalent to (4.6).

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