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TOPOLOGICAL PRODUCT LATTICE

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Abstract: G. Birkhoff studied Topological Lattices. G. Bezhanishvili and Mamuka extended the notion of such a Lattice. Our study is based on the convergence in any open nhd in terms of net on the product of Topological Lattices.

Keyword: Topological Lattice, Open nhd, Convergence

Introduction

A topological lattice is lattice L equipped with a topology T such that meet and join operations from $L \times L$ (with the product topology) to L are continuous.

Convergence: Let $(x_i)_{i\in I}$ be a net in L, we say that (x_i) converges to x if (x_i) is eventually in any open neighbourhood (nhd) of x ,and we write $x_i \to x$.

If (x_i) and (y_i) are net indexed by I, J respectively, then $(x_i \wedge y_i)$ and $(x_i \vee y_i)$ are nets, both indexed by $I \times J$. This is clear, and is stated in preposition below:

Theorem: If $x_i \to x$ and $y_i \to y$, then

$$x_i \wedge y_i \to x \wedge y \text{ and } x_i \vee y_i \to x \vee y$$

Proof: Let us show the first convergence and the other follows similarly. The function

$$f : x \to (x , y) \to x \wedge y$$

is a continuous function , being composition of two function. If $x \wedge y \in U$ is open, then $x \in f^{-1}(U)$ is open. As $x_i \to x$, there is an $i^0 \in I$ such that $x_i \in f(U)$ for all $i \geq i^0$ which means $x_i \wedge y = f(x_i) \in U$. By the same token, for each $i \in I$, the function $g_1 : y \to (x_i\, y) \to x_i \wedge y$ is a continuous function. Since $x_i \wedge y \subset U$ is open, $Y \in g^{-1}(U)$ is open. As $y_i \to y$, there is an $j_o \in J$ such that $y_i \in g^{-1}(U)$ for all $j \geq j_o$, or $x_i \wedge y_i = g_1(y_i) \in U$ for all $i \geq i_o$ and $j \geq j_o$. Hence $x_i \wedge y_i \to x \wedge y$. For any net (x_i), the set A = $\{a \in L \,/\, x_i \to a\}$ is a sublattice. This follows from the fact that if a , $b \in A$, then

$$x_i = x_i \wedge x_i \to a \wedge b. \text{ So } a \wedge b \in A.$$

Similarly $\quad a \vee b \in A.$

There are two approaches to finding examples of topological lattices. One way is to start with a topological space X such that X is partially ordered, then find two continuous binary operations on X to form the meet and join operation of a lattice.

The real numbers with operations $a \wedge b = \inf \{a, b\}$ and $a \vee b = \sup \{a, b\}$ is one such example. [3] This can be easily generalised to the real-valued continuous functions, since, given any two real valued continuous functions f and g

$$f \vee g = \max (f, g) \text{ and } f \wedge g = \min (f, g)$$

are well defined real valued continuous functioned as well (in, it is enough to say they for any continuous function f, its absolute value $| f |$ is also continuous

so that $\quad \max (f, 0) = 1/2 (f + | f |))$ thus $\max (f, g) = \max (f - g, 0) + g$

and $\quad \min (f, g) = f + g - \max (f, g)$ are both continuous)

The second approach is to start with a general lattice L and define a topology T on the subset of the underling set L with the hope that both $\vee$ and $\wedge$ are continuous under T. The obvious example using the second approach is to take the discrete topology of the underlying set. Another way is to impose conditions, such as requiring that the lattice be meet and join continuous. Of course, finding a topology on underlying of a lattice may not guarantee a topological lattice unless and until the lattice operations are continuous.

References

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