A Study of Finsler spaces in Differential Geometry

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ABSTRACT

In a Finsler space with a Finsler metric \( L(X,Y) \), the fact that the fundamental tensor field \( g_{ij}(x,y) \) is provided as \( g_{ij} = \left( \frac{\partial^2 L^2}{\partial Y^i \partial y^j} \right)/2 \) from the fundamental function \( L \) plays an important role in the study of the Finsler space but is said to be not necessarily desirable from the point of view of theoretical physics.

INTRODUCTION

Finsler space is smooth Manifold possessing a finsler Metric. Finsler Geometry is R-Geometry without condition that the line element be quadratic. The purpose of the present paper is to study some properties of generalized Finsler space and then one or our metrical Finsler connections is used. CH-1 is devoted to the preliminaries. In CH-2 we shall consider generalized Finsler spaces corresponding to locally Minkowski spaces. In CH-3, we consider generalized Finsler spacer. Corresponding to Berwald spaces. And in CH-4 we deal with a special metrical Finsler strcture and, in a generalized Finsler space with this special metrical Finsler space with this special metrical Finsler structure, we discuss the results obtained in CH-2 and CH-3

The terminologies and notations in the present paper are referred to Matsumoto’s book [4].
1. PRELIMINARIES

Let us quote the results necessary for the subsequent considerations from Matsumoto’s book [4]. Let \( \Gamma = (F^i_{jk}, N^i_j, C^i_{jk}) \) be a Finsler connection on an \( n \) –dimesional generalized Finsler space \( MN \). Then, with respect to the Finsler connection \( \Gamma \), the h-and v-covariant derivatives of a Finsler tensor field \( T^i_j \), for example, are given by-

\[
T^i_j|k = \delta_k T^i_j + T^h_j F^i_{hk} - T^i_j F^h_{jk},
\]

\[
T^i_j = \delta_k T^i_j + T^h_j C^i_{hk} - T^i_j C^h_{jk},
\]

Where \( \delta_k = \delta_k - N^i_j \delta_i, \delta_k = \delta / \delta x^k \) and \( \delta_k = \delta / \delta y^k \). And, five torsion tensor fields and three curvature tensor fields of the Finsler connection \( \Gamma \) are obtained as follows:

\begin{align*}
(1.1) & \quad C^i_{jk} \quad \text{(h) hv-torsion} \\
(1.2) & \quad R^i_{jk} = A_{jk}\{\delta_k N^i_j \} \quad \text{(v) h-torsion,} \\
(1.3) & \quad T^i_{jk} = A_{jk}\{F^i_{jk}\} \quad \text{(h) h-torsion,} \\
(1.4) & \quad P^i_{jk} = \delta_k N^i_j - F^i_{kj} \quad \text{(v) hv-torsion,} \\
(1.5) & \quad S^i_{jk} = A_{jk}\{C^i_{jk}\} \quad \text{(v) v-torsion,} \\
(1.6) & \quad R^i_{hk} = A_{jk}\{\delta_k F^i_{hk} = F^r_{hk} F^i_{rj}\} + C^i_{hr} R^r_{jk} \quad \text{... h-curvature,} \\
(1.7) & \quad P^i_{hk} = \delta_k F^i_{hk} + C^i_{hki} + C^i_{hr} P^r_{jk} \quad \text{... h v-curvature,} \\
(1.8) & \quad S^i_{hjk} = A_{jk}\{\delta_k C^i_{hk} + C^i_{hr} C^r_{jk}\} \quad \text{... v-curvature}
\end{align*}

Where \( A_{(jk)} \) means the interchange of the indices \( j, k \) in the terms enclosed with the parentheses \( \{ \} \) and subtraction.

Hereafter, we shall deal with the Finsler connection \( F^i_j = (F^i_{jk}, N^i_j, C^i_{jk}) \) for which the following conditions are satisfied:

\begin{align*}
(1.9) & \quad (a) \quad T^i_{jk} = 0, \quad (b) \quad S^i_{jk} = 0, \\
(1.10) & \quad F^i_{jk} y^j = N^i_k, \\
(1.11) & \quad (a) \quad g^i_{ijk} = 0 \quad (b) \quad g^i_{ijk} = 0
\end{align*}

A Finsler connection on \( MN \) is called metrical if it satisfies the conditions (1.11) (a) and (b). In order to use later on, for the above metrical Finsler connection, let us introduce one of the Bianchi identities.

\begin{align*}
(1.12) & \quad P^i_{ijk1} = P^i_{jik1} \\
(1.13) & \quad A_{(jk)}\{C^i_{khj} + C^i_{thr} P^r_{ki} - P^r_{jhi} \} = 0
\end{align*}

Where \( C^i_{khi} = g^i_{hr} C^r_{ki} \) and \( P^r_{khi} = g^i_{hr} P^r_{ik1} \).

Now introduce an identity necessary for the subsequent consideration. Applying he Christoffel process [4] with respect to the indices \( j,k,h \) to the Bianchi identity (1.12) and using (1.13), we have-

\[
2P^i_{jhi} = (C^i_{khi} + C^i_{hki}) - (C^i_{hji} - C^i_{jhi})_{jk} - (C^i_{kji} + C^i_{jki})_{jh} + (C^i_{jhi} - C^i_{hji}) P^r_{ki} + (C^i_{kjr} + C^i_{jkr}) P^r_{hi} - (C^i_{khr} + C^i_{hkr}) P^r_{ji}.
\]
Again we are going to determine a metrical Finsler connection $\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ with the conditions (1.9) and (1.10). then, it is easily seen that—

(1.15) $C^i_{jk} = g^{ih}(\delta_j g_{hk} + \delta_k g_{hi} - \delta_h g_{jk})/2$

Let $(A^i_j)$ be a matrix defined by—

(1.16) $A^i_j = \delta^i_j + g^{ir} y^s \delta_i g_{rs}$

Assuming that the matrix $(A^i_j)$ is regular, we can consider a matrix $(A^{pq}_{kj})$, (pq, kj = 11,12, ..., In, ..., n1, n2 ..., nn), defined by

(1.17) $A^{pq}_{kj} = 2\delta^q_k \delta^p_i + \delta^q_k g^{pr} y^i \delta_r g_{ij} + B^r_{rs} C_{soo} g^{so} g^{qr} \delta_i g_{jk} - g^{rp} y^i \delta^q_r \delta_r g_{ki}$

Here the index $o$ denotes the contraction by y and the matrix $(B^r_{rs})$ is the inverse matrix of $(A^i_j)$.

**THEOREM 1:**

Let $M^n$ be a generalized Finsler space and let us assume that two matrices $(A^i_j)$ and $(A^{pq}_{kj})$ are regular. Then, there exists uniquely a metrical Finsler connection $\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ with the conditions (1.9) and (1.10).

It is given by (1.15), (1.10) and (1.18)

(1.18) $F^i_{jk} = \gamma^i_{jk} - g^{ir}(F^s_{ok} \delta_s g_{kr} - F_{osr} \delta_r g_{jk})/2$, where

(1.19) $F_{osr} = g_{rj} \gamma^i_{jk} = (\delta^i_r g_{ik} + \delta_k g_{ji} - \delta_j g_{ik})/2$

(1.20) $\gamma^i_{ijk} = g_{rj} \gamma^r_{ijk} = (\delta^i_r g_{ik} + \delta_k g_{ji} - \delta_j g_{ik})/2$

And the matrix $(B^r_{rs})$ is the inverse matrix of $(A_{kj}^{pq})$.

In the subsequent considerations, we use the Finsler connection given by Theorem I.

**Remarks I.** In Finsler spaces, the metrical Finsler connection in theorem 1 reduces to the Cartan connection.

**II. Existence of a coordinate system (xi) in which gij are functions of ys only.**

In this section, we shall prove the following theorem.

**THEOREM 2**

A necessary and sufficient condition that there exist a coordinate system $(x^i)$ for a generalized Finsler space $M^n$ for which the components of the metrical Finsler structure $g_{ij}$ are function of $y_i$ only is that

(2.1) $R_{ijk} = 0$
(2.2) $R_{hijk} = 0$
(2.3) $C_{hijkl} = 0$
(2.4) $P_{rjk} = 0$

Where $P_{ijk} = g^{ir} P^r_{jk}$
PROOF:

Assume that $g_{ij}$ are functions of $y^i$ only in a coordinate system $(x^i)$. Then, using (1.18), (1.19) and (1.20), we have $F_{jk}^i = 0$ and then $N_k^i = F_{j[k}^i y^{i]} = 0$. From this equation and (1.2), we obtain (2.1). Using (1.6), (2.2) results at once from (2.1) and the fact that $F_{jk}^i = 0$. Using (1.15), the assumption gives that $C_{jk}^i$ are functions of $y^i$ only. Accordingly, using $F_{jk}^i = 0$ and $N_j^i = 0$, we have (2.3), (2.4) is obvious b (1.4).

Next, we shall prove that converse. From (2.3) and (2.4), (1.14) gives $P_{jhki} = 0$. From this equation, (1.7), (2.3) and (2.4), we have $\delta_k F_{hj}^i = 0$, that is, $F_{hj}^i$ are functions of position only. Accordingly, noticing (2.1) and (2.2), we have that (1.6) reduces to-

$$A_{(jk)} \{ \delta_k F_{hj}^i + F_{hj}^r F_{rk}^i \} = 0.$$

From these facts, in the similar way to the case of a Riemannian space with zero curvature, we get that there exists a coordinate system $(\overline{x}^a)$ such that $F_{bc}^a = 0$. Hence we have $N_k^a = 0$, so that (1.11) (a) gives that $\overline{g}_{ab}$ are functions of $y^{-a}$ only Q.E.D

Remark 2. For the Cartan connection in a Finsler space, (2.1) and (2.2) implies because of the identity $R_{hijk} y^h = R_{ijkl}$ and (2.3) implies (2.4) because $C_{ijk|o} = P_{ijk}$. But in our case, the above stated facts are noe true, since we have that $R_{hlik} y^h = R_{ijkl} + C_{oir} R_{rk}^i$ and tht $C_{ijk|o} = P_{ijk}$ is not necessarily satisfied.

2. A condition for $F_{jk}^i$ to be functions of position only.
3. In this section we shall prove the following theorem.

**THEOREM 3**

A necessary and sufficient condition for the connection coefficients $F_{jk}^i$ to be functions of position only in that the following equations are satisfied.

1. $P_{ijk} = 0$,
2. $C_{khi|j} + C_{hki|j} = 0$
**PROOF:**

The fact that \( F_{jk}^i \) are functions of position only is equivalent to a conditions \( \delta_h F_{jk}^i = 0 \). And, from (1.7), the conditions

\[
\delta_h F_{jk}^i = 0
\]

is equivalent to

\[
(3.3) p_{hijk} = -c_{hij}^k + c_{hik}^j p_{jk}^i
\]

Now we assume that \( \delta_h F_{jk}^i = 0 \). Then,

From (1.4), we have \( p_{jk}^i = 0 \). from this and (3.3), we have

\[
(3.4) p_{hjki} = -c_{hjik}
\]

Noticing \( p_{jk}^i = 0 \), from (1.14) we have

\[
(3.5) 2p_{hjki} = c_{jik}^h + c_{hik}^j - c_{hij}^k + c_{hj}^k - c_{kh}^i - c_{hi}^j + c_{h}^i - c_{hj}^k = 0
\]

From (3.4) and (3.5), we have

\[
(3.6) c_{kij}^h + c_{jki}^h - c_{khi}^l - c_{hki}^j - c_{hjl}^i + c_{jli}^h = 0
\]

Cyclic permutations of indices \( h, j, k \) in (3.6) and summation yield

\[
(3.7) c_{kij}^l + c_{jki}^l + c_{khi}^l + c_{hki}^j + c_{hjl}^i + c_{jli}^h = 0
\]

From (3.6) and (3.7), we have (3.2)

Next we shall prove the converse. Substituting (3.1) and (3.2) into (1.14), we have

\[
2p_{jik} = c_{hji}^k - c_{jhi}^k = -2c_{jh}^k, \text{ thus we have (3.3) equivalent to the condition } \delta_h F_{jk}^i = 0.
\]

Q.E.D

Remarks 3. For the cartan connection in a Finsler space, (3.2) implies (3.1).

4. A generalized Finsler space with

\[
g_{ij} = e^{2\sigma(x,y)} y_{ij}(x)
\]

In this section, we shall consider a generalized Finsler space Mn with a special metrical Finsler structure.

\[
g_{ij} = e^{2\sigma(x,y)} y_{ij}(x)
\]

Where \( \sigma \) is positively homogeneous of degree 0 with respect to \( y \) and \( Yij \) is Ricmannian metric of Mn. Then, the following theorem is known.
THEOREM 4
[5], [6], Let \( M^n \) be a generalized Finsler space with \( g_{ij} \) given by (4.1). Then, under the conditions (1.9) and (1.10), there exists uniquely a metrical Finsler connection
\[
F \Gamma = \left( F^i_{jk}, N^i_k, C^i_{jk} \right)
\]
on \( M^n \).
From (1.15), we get easily
\[
(4.2) \quad C_{ijk} = \left( C_i g_{uk} + C_k g_{ji} - C_{jik} \right)/n
\]
\[
(4.3) \quad C_j = C_{iij} = n \delta_i^\sigma
\]

THEOREM 5
With respect to the metrical Finsler connection \( F \Gamma \) in theorem 4, the following three conditions are equivalent
\[
(4.4) \quad C_{ijk|h} + C_{ijk|h} = 0
\]
\[
(4.5) \quad C_{ijk|h} = 0
\]
\[
(4.6) \quad C_{i|h} = 0
\]

PROOF:
Differentiating (4.2) \( h \)-covariantly with respect to \( x^h \) we have
\[
(4.7) \quad C_{ijk|h} = \left( C_{i|h} g_{jk} + C_k g_{ji} - C_{jik|h} \right)/n
\]
From this equation, we get \( C_{ijk|h} + C_{k|h} = 2\delta_{k|h} g_{ij} \). Thus, (4.4) is equivalent to (4.6).
Using (4.7), we have easily that (4.5) is equivalent to (4.6).

From theorem 5, in the case of the generalized Finsler space considered in this section, we have that the condition (2.3) (resp. (3.2) in Theorem 2 (resp. Theorem 3) is equivalent to (4.6).

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