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Iterative Methods for Solving Ordinary Differential Equations: A Review

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Abstract

In the present paper we review some iterative methods for solving initial value problems of ordinary differential equations. The numerical methods are compared in terms of their convergence, accuracy and efficiency. Differential equations are one of the most important mathematical tools used to model physical and biological processes. Numerical methods are an important part of solving differential equations that arises from real-life situations, most especially in cases when it is difficult to obtain exact solutions by conventional methods.

Keywords: Initial Value Problem, Consistency; Stability and convergence; Order of methods, Runge-Kutta method.

1. Introduction

In recent years, a large number of iterative methods suitable for solving differential equations has been proposed. The approach for solving differential equations based on numerical approximations was developed before the existence of programmable computers. It is the province of numerical analysis to study and implementation of such methods. The development of numerical methods for the solution of initial value problems in ordinary differential equations have attracted the attention of many researchers in recent years. Notable are the 1883 paper of Bashforth and Adams [2] and the 1895 paper of Runge [3]. These two papers generate the general initial value problem in much the same form. That is, given a function $f(x, y)$ and the initial value y_0 corresponding to a solution at x_0, we try to evaluate numerically the function y satisfying

$$y'(x) = f\big(x, y(x)\big), \qquad y(x_0) = y_0 \tag{1}$$

The approach is to extend the set of x values step-by-step for which an approximation to $y(x)$ is known.

2. Preliminaries

In this section we discuss some of the basic definitions and results of numerical analysis [7-8].

Definition 2.1

The difference between exact solution $y(x_j)$ at $x = x_j$ and the solution y_j determined from numerical method is called local truncation error. We have,

$$\|T_j\| = y(x_j) - y_j \tag{2}$$

Definition 2.2

If the cumulative effect of all errors, including round-off error is bounded, independent of the number of mesh points then a method is called stable.

Definition 2.3

A general single-step method can be written in the form

$$y_{j+1} - y_j = h\phi(x_j, y_j, h) \tag{3}$$

Definition 2.4

The largest integer p such that

$$\|h^{-1}T_j\| = O(h^p) \tag{4}$$

Definition 2.5

A single step numerical method (3) is said to be consistent if

$$\phi(x, y, 0) = f(x, y) \tag{5}$$

Definition 2.6

A single step method (3) is said to be regular if the function $\phi(x, y, h)$ is defined and continuous in the domain $a \leq x \leq b,\ -\infty < y^i < \infty, i = 1,2,3, \ldots, n, 0 \leq h \leq h_0$ and if there exist a constant L such that

$$\|\phi(x, y, h) - \phi(x, y^*, h)\| \leq L\|y - y^*\| \tag{6}$$

Theorem 2.7

Suppose the single step method (3) is regular. Then the relation (5) is a necessary and sufficient condition for the convergence of the method defined by (3).

3. The theory of Runge-Kutta methods.

Following the important work of Runge and of Adams , the further contributions to what is now known as the Runge-Kutta method, by Heun [1] and Kutta [5]. Runge-Kutta methods are one step in the view that result computed at the end of step is dependent only on the result given at the end of previous step. So, if y_n is a calculated approximation to $y(x_n)$, then y_n will we of the form

$$y_n = y_{n-1} + h\sum_{i=1}^{k} b_i G_i \ ,$$

Where the terms $G_i\ , i = 1,2, \ldots . . k$ are the derivatives derived from approximations $Y_i\ , i = 1,2, \ldots . . k$ to the solutions at $x_{n-1} + h\, c_i\ , i = 1,2, \ldots . . k$. that is,

$G_i = f(\, x_{n-1} + h\, c_i\, , Y_i\,) \quad , i = 1,2, \ldots . . k$

For the differential equation (1). The values of $Y_i\ , i = 1,2, \ldots . . k$ are computed from the equation

$$y_i = y_{n-1} + h\sum_{j=1}^{k} a_{ij} G_j \ , \quad i = 1,2, \ldots . . k\ .$$

So, the components of the vector c are related to the elements of the matrix A by,

$$c_i = \sum_{j=1}^{k} a_{ij} \ , \quad i = 1,2, \ldots . . k\ .$$

The number of iterations k is the number of vectors Y needed to compute the solution in this form of method.

The characteristic coefficients of Runge-Kutta method are displayed in following table

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \ldots & a_{1k} \\
c_2 & a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \vdots & & \vdots \\
c_k & a_{k1} & a_{k2} & \ldots & a_{kk} \\
\hline
 & b_1 & b_2 & \ldots & b_k
\end{array}$$

The fourth order method proposed by Kutta became very popular and it is entitled as 'The Runge-Kutta method'.

For verification of order of these and other Runge-Kutta methods it is necessary to expand the computed and exact solutions in powers of h and to check that the terms up to and including those with an exponent p agree with each other. It can be proved that the exact solution for (1) near an initial point (x_0, y_0) is given by expansion

$$y(x_0 + h) = \ y_0 + h\boldsymbol{f} + \frac{h^2}{2}\boldsymbol{f}'(\boldsymbol{f}) + \frac{h^3}{6}(\boldsymbol{f}''(\boldsymbol{f},\boldsymbol{f}) + \boldsymbol{f}'(\boldsymbol{f}'(\boldsymbol{f}))) + \ \frac{h^4}{24}(\boldsymbol{f}'''(\boldsymbol{f},\boldsymbol{f},\boldsymbol{f}) + 3(\boldsymbol{f}''(\boldsymbol{f},\ \boldsymbol{f}'(\boldsymbol{f})) + \boldsymbol{f}'(\boldsymbol{f}''(\boldsymbol{f},\boldsymbol{f})) + \boldsymbol{f}'(\boldsymbol{f}'(\boldsymbol{f}'(\boldsymbol{f})))) + O(h^5), \qquad (7)$$

where $\boldsymbol{f} = f(y_o)$, $\boldsymbol{f}' = f'(y_o)$, $\boldsymbol{f}'' = f''(y_o)$, $\boldsymbol{f}''' = f'''(y_o)$

on other side, the Taylor series expansion of the numerical solution computed by an explicit Runge-Kutta method with $k = 4$ is equals to

$$y_1 = y_0 + h(b_1 + b_2 + b_3 + b_4)\boldsymbol{f} + \frac{h^2}{2}(b_2c_2 + b_3c_3 + b_4c_4)\boldsymbol{f}'(\boldsymbol{f})$$
$$+\frac{h^3}{6}\Big(2\big(b_2c_2^2 + b_3c_3^2 + b_4c_4^2\big)\boldsymbol{f}''(\boldsymbol{f},\boldsymbol{f}) + (b_3a_{32}c_2 + b_4a_{42}c_2 + b_4a_{43}c_3)\boldsymbol{f}'(\boldsymbol{f}'(\boldsymbol{f}))\Big)$$
$$+\frac{h^4}{24}(6(b_2c_2^3 + b_3c_3^3 + b_4c_4^3)\boldsymbol{f}'''(\boldsymbol{f},\boldsymbol{f},\boldsymbol{f}) + (b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 +$$
$$b_4a_{43}c_3c_4)\boldsymbol{f}''\big(\boldsymbol{f},\boldsymbol{f}'(\boldsymbol{f})\big)$$
$$+(b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{42}c_3^2)\boldsymbol{f}'\big(\boldsymbol{f}''(\boldsymbol{f},\boldsymbol{f})\big)$$
$$+b_4a_{43}a_{32}c_2\boldsymbol{f}'(\boldsymbol{f}'(\boldsymbol{f}'(\boldsymbol{f})))) + O(h^5) \tag{8}$$

By Comparing (7) and (8) shows that agreement up to h^4 terms hold iff

$$b_1 + b_2 + b_3 + b_4 = 1,$$

$$b_2c_2 + b_3c_3 + b_4c_4 = \frac{1}{2}\ ,$$

$$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \frac{1}{3}$$

$$b_3a_{32}c_2 + b_4a_{42}c_2 + b_4a_{43}c_3 = \frac{1}{6}$$

$$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = \frac{1}{4}$$

$$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 = \frac{1}{8}$$

$$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{42}c_3^2 = \frac{1}{12}$$

$$b_4a_{43}a_{32}c_2 = \frac{1}{24}$$

To get the classical fourth order method, it is easily seen to satisfied by the values

$$a_{21} = a_{32} = c_2 = c_3 = \frac{1}{2}\ ,$$

$$a_{31} = a_{41} = a_{42} = 0\ ,$$

$$a_{43} = c_4 = 1\ ,$$

$$b_1 = b_4 = \frac{1}{6},$$

$$b_2 = b_3 = \frac{1}{3}\ .$$

Explicit methods have as high as order 10 but this tends to increasingly more stages. For a reference, for $p = 5, k = 6$ is necessary and for $p = 8, k = 11$ is required. For Implicit methods relation between p and

k is simple as compare to explicit methods. That is $\forall\, k \in \mathbf{Z}^+$ there is an implicit Runge-Kutta Method having order $p = 2k$ (but not higher).

4. Method based on Taylor series.

J. Sunday et.al. [4] developed one-step scheme for the solution of initial value problems of first order in ordinary differential equations by using combination of interpolating function and Taylor series. The numerical discussion begins by interpolating power series

$$y(x) = \sum_{i=1}^{1} \alpha_i x^i + \alpha_2 e^{-x} \tag{9}$$

With integration interval of $[a, b]$ in the form $a = x_0 < x_1 <\ldots< x_n < x_{n+1} <\ldots< x_N = b$ with step size h given by $h = x_{n+1} - x_n$ such that $n = 1,2, \ldots\ldots, N - 1$.

Which yields to

$$y_{n+1} - y_n = hf_n + [h + (e^{-h} - 1)]f_n' \tag{10}$$

5. Conclusion:

In this short review of iterative methods for ordinary differential equations, it has been possible to mention only a few aspects of this very active research area.

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