IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

NORMAL FUZZY SUBGROUP & CHARACTERISTIC FUZZY SUBGRUOP

Amit Kumar Arya

Research Scholar, P.G. Department of Mathematics, M. U. Bodh Gaya, Bihar

Abstract

in this paper, I search of the properties of normal fuzzy subgroup. We also introduced the notation of characteristic fuzzy subgroup and some result of homomorphism of subgroup. Then we prove that characteristic fuzzy subgroup is also normal fuzzy subgroup. The characteristic of fuzzy subgroup [13] was first introduced by P. Bhattacharya and N.P. Mukharjee in 1986.

Keywords : Fuzzy subgroup, characteristic fuzzy subgroup, abelian fuzzy subgroup and normal fuzzy subgroup.

1. Introduction

The concept of fuzzy sets was introduced by L.A.Zadeh [15] in 1965.Study of algebraic structure was first introduced by A. Rosenfeld [1]. After that a series of researches have done in this direction P. Bhattacharya and N.P. Mukharjee [13] have defined fuzzy normal subgroup and characteristic fuzzy subgroup in 1986. In this paper we have tried to established some independent proof about the properties of fuzzy normal subgroup and characteristic fuzzy subgroup.

2. Preliminaries

In this section, we study some concepts associated with fuzzy sets and fuzzy group, normal fuzzy subgroup and characteristic fuzzy subgroup. In this section, we recall some basic definitions and results which will be used in the Fuzzy Inf. Eng. (2012)

2.1 Fuzzy Set

Over the past three decades, a number of definitions of a fuzzy set and fuzzy group have appeared in the literature (cf., e.g., [15, 1, 3, 7, 10]). In [15], it has been shown that some of these are equivalent. We begin with the following basic concepts of fuzzy set, fuzzy point and fuzzy group.

Definition 2.1 [15] **fuzzy power set**. A fuzzy subset of D_1 be a function $f_1 : D_1 \rightarrow [0,1]$ the set of all fuzzy subset of D_1 is sad to be fuzzy power set of D_1 and designate by $P_1(D_1)$.

Definition 2.2 [15] **Support of fuzzy set**. Suppose $A_1 \in F_1\ P_1(D_1)$ then the set $\{\ A_1\ (d_1) : d_1 \in D_1\}$ is said to be the image of A_1 is designate by $A_1\ (D_1)$. The set $\{d_1 : d_1 \in D_1\ ,\ A_1\ (d_1) > 0\}$ is said to be the support of A_1 is designate by A_1*.

Definition 2.3 [15] Let $A_1, C_1 \in F_1\ P_1(D_1)$ such that $A_1\ (d_1) \leq C_1\ (d_1), \forall\ d_1 \in D_1$ then A_1 is said to be contained in C_1 and it is designate by $A_1 \subseteq C_1$

Definition 2.4 [15] Let $B_1 \subseteq A_1$ and $d_1 \in [0,1]$ we defined $d_{1_{B_1}} \in F_1\ P_1(D_1)$ as

$$d_{1_{C_1}}\ (\text{a}) = \begin{cases} d_1, for\ a_1 \in B_1 \\ 0,\ for\ a_1 \in A_1 \end{cases}$$

If B_1 is a singleton $\{b_1\}$ then $D_{\{b_1\}}$ is called a fuzzy point.

For any collection $\{A_{i_1}, i_1 \in I_1\}$ of fuzzy subset of D_1, where I_1 is an index set the least upper bound (L.U.B.) $\bigcup_{i_1 \in I_1} A_{i_1}$ and greatest lower bound (G.L.B) $\bigcap_{i_1 \in I_1} A_{i_1}$ of A_{i_1} are given by

$$(\bigcup_{i_1 \in I_1} A_{i_1})\ (d_1) = \vee_{i_1 \in I_1} A_{i_1}\ (d_1), \quad \forall\ d_1 \in D_1.$$

$$(\bigcap_{i_1 \in I_1} A_{i_1})\ (d_1) = \wedge_{\ i_1 \in I_1} A_{i_1}\ (d_1), \quad \forall\ d_1 \in D_1$$

3. NORMAL FUZZY SUBGROUP [18, 19, 20]

In this section we study of normal fuzzy subgroup and established some independent proof.

Proposition 3.1: Let $\mathrm{A}_1 \in \mathrm{F}_1\mathrm{P}_1\ (\mathrm{G}_1)$ then the succeeding statements are equivalent.

i. $\mathrm{A}_1\ (d_1c_1) = \mathrm{A}_1\ (c_1d_1), \quad \forall\ d_1,\ c_1 \in \mathrm{G}_1$

ii. $\mathrm{A}_1(d_1c_1{d_1}^{-1}) = \mathrm{A}_1\ (c_1), \quad \forall\ d_1,\ c_1 \in \mathrm{G}_1$

iii. $\mathrm{A}_1(d_1c_1{d_1}^{-1}) \geq \mathrm{A}_1\ (c_1), \quad \forall\ d_1,\ c_1 \in \mathrm{G}_1$

iv. $\mathrm{A}_1(d_1c_1{d_1}^{-1}) \leq \mathrm{A}_1\ (c_1), \quad \forall\ d_1,\ c_1 \in \mathrm{G}_1$

v. $\mathrm{A}_1 \text{ o } \mathrm{B}_1 = \mathrm{B}_1 \text{ o } \mathrm{A}_1, \ \forall\ \mathrm{B}_1 \in \mathrm{F}_1\mathrm{P}_1(\mathrm{G}_1)$

Proof: (i) ⇒ (ii) if $\mathrm{A}_1\ (d_1c_1) = \mathrm{A}_1\ (c_1d_1), \quad \forall\ d_1,\ c_1 \in \mathrm{G}_1$

Then, $\mathrm{A}_1(d_1c_1{d_1}^{-1}) = \mathrm{A}_1(c_1d_1{d_1}^{-1}) = \mathrm{A}_1\ (c_1), \quad \forall\ d_1,\ c_1 \in \mathrm{G}_1$

(ii) ⇒ (iii) $\mathrm{A}_1(d_1c_1{d_1}^{-1}) \geq \mathrm{A}_1\ (c_1)$ obviously

(iii) ⇒ (iv) $\mathrm{A}_1(d_1c_1{d_1}^{-1}) \leq \mathrm{A}_1\ ({d_1}^{-1}d_1c_1{d_1}^{-1}({d_1}^{-1})^{-1}) = \mathrm{A}_1c_1, \forall\ d_1, c_1 \in \mathrm{G}_1$

(iv) ⇒ (i)

$$\begin{aligned} \mathrm{A}_1\ (d_1c_1) &= \mathrm{A}_1\ (d_1c_1d_1{d_1}^{-1}) \\ &= \mathrm{A}_1(d_1(c_1d_1){d_1}^{-1}) \\ &\leq \mathrm{A}_1\ (c_1d_1) \\ &= \mathrm{A}_1\ (c_1d_1c_1{c_1}^{-1}) \\ &= \mathrm{A}_1\ (c_1\ (d_1c_1){c_1}^{-1}) \\ &\leq \mathrm{A}_1\ (d_1c_1) \end{aligned}$$

Finally, $\mathrm{A}_1\ (d_1c_1) = \mathrm{A}_1\ (c_1d_1)$

(i) ⇒ (v) Let $d_1 \in \mathrm{G}_1$ then

$$(A_1 \text{ o } B_1)(d_1) = \vee_{c_1 \in G_1} \{A_1(d_1 c_1^{-1}) \wedge B_1(c_1)\}$$
$$= \vee_{c_1 \in G_1} \{B_1(c_1^{-1} d_1) \wedge B_1(c_1)\}$$
$$= (B_1 \text{ o } A_1)(c_1), \qquad \forall d_1 \in G_1$$

Finally, $(A_1 \text{ o } B_1) = (B_1 \text{ o } A_1)$

(v) ⇒ (i) We have

$$1(c_1^{-1}) \text{ o } A_1 = A_1 \text{ o } 1(c_1^{-1}), \qquad \forall c_1, d_1 \in G_1$$

Thus $(1(c_1^{-1}) \text{ o } A_1)(d_1) = \big(A_1 \text{ o } (c_1^{-1})\big)(d_1), \qquad \forall d_1, c_1 \in G_1$

Hence $A_1(d_1 c_1) = A_1(c_1 d_1), \qquad \forall d_1, c_1 \in G_1$

Proposition 3.2: Suppose $A_1 \in F_1(G_1)$. Then ${A_1}_*$, ${A_1}^*$ are normal subgroup of G_1.

Proof: Since we know

$${A_1}_* = \{d_1 \in G_1 : A_1(d_1) = A_1(e)\}$$

And

$${A_1}^* = \{d_1 \in G_1 : A_1(d_1) > 0\}$$

Since $A_1 \in F_1(G_1)$, it follows form the use of lemma that ${A_1}_*$, ${A_1}^*$ are subgroup of G_1.

Occupancy $d_1 \in G_1$, and $c_1 \in {A_1}_*$

SinceA_1 is satisfy the condition

$$A_1\big(d_1 c_1 d_1^{-1}\big) = A_1(c_1) = A_1(e)$$

Hence, $(d_1 c_1 d_1^{-1}) \in {A_1}_*$

Hence, $(d_1 c_1 d_1^{-1}) \in {A_1}_*$

Also let $d_1 \in G_1$ and $c_1 \in {A_1}^*$

$\Rightarrow A_1(c_1) > 0$.

Since A_1 satisfy the condition $A_1\big(d_1 c_1 d_1^{-1}\big) \geq A_1(c_1) > 0$

Implies that $\big(d_1 c_1 d_1^{-1}\big) \in {A_1}^*$

Therefore, ${A_1}^*$is a normal subgroup of G_1.

Proposition 3.3: Suppose $A_1 \in F_1(G_1)$, let $N_1(A_1) = \{d_1 : d_1 \in G_1, A_1(d_1 c_1) = A_1(c_1 d_1) \, \forall c_1 \in G_1\}$ Formerly $N_1(A_1)$ be a subgroup of G_1 and the limit of A_1 to $N_1(A_1)$, $A_1 \mid N_1(A_1)$ is a normal fuzzy subgroup of $N_1(A_1)$

proof : Suppose $d_1, c_1 \in N_1(A_1)$, $e \in N_1(A_1)$

For any $t_1 \in G_1$, we have

$$A_1(d_1 c_1^{-1} t_1) = A_1(d_1 c_1^{-1} t_1)$$
$$= A_1(c_1^{-1} t_1 d_1)$$
$$= A_1(c_1^{-1} t_1 d_1)^{-1} \therefore \quad A_1(d_1) = A_1(d_1^{-1})$$
$$= A_1\big(d_1^{-1} t_1^{-1} c_1\big) \text{ as } A_1 \in F_1(G_1)$$

$$= A_1(c_1\ (t_1 d_1)^{-1})$$
$$= A_1(c_1(t_1 d_1)^{-1})^{-1}$$
$$= A_1\ (t_1 d_1 c_1{}^{-1})$$
$$\Rightarrow \quad d_1\, c_1{}^{-1} \in N_1\ (A_1)$$

Later $N_1\ (A_1)$ is a subgroup of G_1,

Since, $A_1 \in F_1\ (G_1)$

Then $A_1 \mid G_2 \in F_1\ (G_2)$

Hereafter $A_1 \mid N_1\ (A_1) \in F_1\ (N_1\ (A_1))$

And $A_1 | N_1(A_1)\ (d_1\, c_1) = A_1 N_1\ (A_1)\ (c_1\, d_1),\ \forall\, c_1, d_1 \in N_1\ (A_1)$

Therefore $A_1 \mid N_1\ (A_1) \in N_1 F_1\ (N_1\ (A_1))$

4. Characteristic fuzzy subgroup [13]

In this section the study of characteristic fuzzy subgroup and normal fuzzy subgroup. We also extend use of normal fuzzy subgroup to characteristic fuzzy subgroup and some independent proof.

DEFINITION 4.1: Let A_1 be a fuzzy subgroup of G_1 and ϕ be a function from G_1 into itself. Now define the fuzzy subset $A_1{}^{\phi}$ of G_1 by $A_1{}^{\phi}(d_1) = A_1\left(d_1{}^{\phi}\right)$, where $d_1{}^{\phi} = \phi(d_1)$ A_1 subgroup K of group G_1 is called a characteristic subgroup if $K^{\phi} = K$ for every automorphism ϕ of G_1, where K^{ϕ}denote $\phi(k)$.

Definition 4.2 Characteristic fuzzy subgroup: A fuzzy subgroup A_1 on a group K is called a fuzzy characteristic subgroup of G_1 if $A_1{}^{\phi}(d_1) = A_1(d_1)$ for every automorphism ϕ of G_1 and for all $d_1 \in G_1$

Proposition 4.3 : Let A_1 is a fuzzy subgroup of a group G_1 if

a. If ϕ is a homomorphism of G_1 into itself, then $A_1{}^{\phi}$ is a fuzzy subgroup of G_1

b. If A_1 is a fuzzy characteristic subgroup of G_1 then A_1 is a normal.

PROOF (i) $d_1, c_1 \in G_1$ then

$$A_1{}^{\phi}(d_1\ c_1) = A_1\ (d_1\ c_1)^{\phi}$$
$$= A_1\ (d_1{}^{\phi} c_1{}^{\phi})$$

Subsequently ϕ is a homomorphism and A_1 is a fuzzy subgroup of G_1.

$$A_1\left(d_1{}^{\phi} c_1{}^{\phi}\right) \geq A_1\left(d_1{}^{\phi}\right) \wedge A_1\left(c_1{}^{\phi}\right)$$

$$A_1{}^{\phi}(d_1\ c_1) = A_1{}^{\phi}(d_1) \wedge A_1{}^{\phi}(c_1)$$

Also, $$A_1{}^{\phi}\left(d_1{}^{-1}\right) = A_1\left(d_1{}^{-1}\right)^{\phi}$$

$$= A_1\left(d_1{}^{\phi}\right)^{-1}$$

$= A_1(d_1{}^{\phi})$

$= A_1{}^{\phi}(d_1)$

Hence, $A_1{}^{\phi}$ is a fuzzy subgroup of G_1. **(ii)** Let d_1, $c_1 \in G_1$ to prove that A_1 is normal we have to show

$A_1(d_1\ c_1) = A_1(c_1\ d_1)$

Let ϕ be function from G_1 into itself definition by $\phi(z) = d_1{}^{-1} z\, d_1$, $\forall\ z \in G_1$ Since A_1 is a fuzzy characteristic subgroup of G_1,

$\therefore A_1{}^{\phi} = A_1$

Thus $A_1(d_1\ c_1) = A_1{}^{\phi}(d_1\ c_1)$

$= A_1(d_1\ c_1)^{\phi}$

$= A_1(\phi(d_1\ c_1))$

$= A_1(d_1{}^{-1}(d_1\ c_1)d_1)$

$= A_1(c_1\ d_1)$

Hence A_1 is normal subgroup of G_1.

Proposition 4.4: Suppose A_1 be a fuzzy normal subgroup of G_1 and Let ϕ be a homomorphism of G_1 into itself then ϕ indues a homomorphism $\bar{\phi}$ of G_1/A_1 into itself defined by

$$\bar{\phi}(d_1A_1) = \phi(d_1)A_1,\ d_1 \in G_1$$

Proof : Suppose $d_1, c_1 \in G_1$, we have

$d_1A_1 = c_1A_1$

$\phi(d_1)A_1 = \phi(c_1)A_1$

Since,

$d_1A_1 = c_1A_1$

We have,

$d_1A_1(d_1) = c_1A_1(d_1)$

$\Rightarrow \quad A_1(e) = A_1(c_1{}^{-1}d_1)$

Also, $d_1A_1(c_1) = c_1A_1(c_1)$

$\Rightarrow A_1(d_1{}^{-1}c_1) = A_1(e)$ $\Rightarrow A_1(c_1{}^{-1}d_1) = A_1(d_1{}^{-1}c_1) = A_1(e)$

$\Rightarrow (c_1{}^{-1}d_1), (d_1{}^{-1}c_1) \in A_{1_*}$ where $A_{1_*} = \{d_1 \in G_1 : A_1(d_1) = A_1(e)\}$

Since we have,

$$\phi\,(A_{1_*}) = A_{1_*}$$

$\phi\,(c_1{}^{-1}d_1)$ and $\phi\,(d_1{}^{-1}c_1)$ also belongs to A_{1_*}

$\Rightarrow A_1\,(\phi\,(c_1{}^{-1}d_1)) = A_1\,(\phi\,(d_1{}^{-1}c_1)) = A_1\,(e)$

Let $g_1 \in G_1$ formerly

$$\phi\,(d_1)A_1(g_1) = A_1\,(\phi\,(d_1{}^{-1})g_1)$$
$$= A_1\,(\phi\left(d_1{}^{-1}\right)\phi\,(c_1)\,\phi\left(d_1{}^{-1}\right)g_1)$$
$$\geq A_1\,(\phi\left(d_1{}^{-1}\right)\phi\,(c_1) \wedge A_1\left(\phi\left(d_1{}^{-1}\right)g_1\right)$$
$$= A_1\,(\phi\left(d_1{}^{-1}d_1\right)) \wedge \phi\,(d_1)\,A_1(g_1)$$
$$= A_1\,(e) \wedge \phi\,(c_1) \wedge A_1(g_1)$$
$$= \phi\,(c_1)\,A_1(g_1)$$

Hence, $(d_1)A_1(g_1) \geq \phi\,(c_1)A_1(g_1)$ ---------------- (i)

Similarly,

$(d_1)\,A_1(g_1) \leq \phi\,(c_1)\,A_1(g_1)$----------------(ii)

Since $g_1 \in G_1$ is arbitrary

Finally, $\phi\,(d_1)A_1 = \phi\,(c_1)A_1$

Therefore, we find that $\overline{\phi}$ is well defined now we have only to show that $\overline{\phi}$ is a

homomorphism.

Let $d_1, c_1 \in G_1$.

Later ϕ is a homomorphism

$$\phi\,(d_1c_1) = \phi(d_1)\,\phi(c_1).$$
$$\phi\,(d_1c_1)\,A_1 = \phi\,(d_1)\,\phi(c_1)A_1$$
$$\Rightarrow \overline{\phi}\,(d_1c_1A_1) = \phi\,(d_1)\,A \text{ o } \phi\,(c_1)A_1$$
$$\overline{\phi}(d_1A_1 \text{ o } c_1A_1) = \overline{\phi}\,(d_1c_1) \text{ o } \overline{\phi}\,(c_1d_1)$$

Finally, $\overline{\phi}$ is a homomorphism.

Proposition 4.5: If A_1, C_1 are fuzzy characteristic subgroup of G_1 then A_1 and C_1 are normal to show

$(A_1 \cap C_1)$ & $(A_1 \cup C_1)$ is too normal.

Proof: Suppose $d_1, c_1 \in F_1(G_1)$to prove that $(A_1 \cap C_1)$ is a normal fuzzy subgroup of G_1 it is necessary to show

$$(A_1 \cap C_1)(d_1\,c_1) = (A_1 \cap C_1)(c_1\,d_1)$$

Let ϕ be the function of group G_1 into itself defined by

$$\phi\,(z) = d_1^{\,-1}\, z\, d_1 \quad \forall\ d_1 \in G_1$$

Since A_1 and C_1 are fuzzy characteristic subgroup of G_1, hence be normal as we prove

$(A_1 \cap C_1)^{\phi} = (A_1 \cap C_1)$

$$(A_1 \cap C_1)(d_1\, c_1) = (A_1 \cap C_1)^{\phi}(d_1\, c_1)$$

$$= (A_1 \cap C_1)(d_1\, c_1)^{\phi}$$

$$= (A_1 \cap C_1)\left(d_1^{\,-1}(d_1\, c_1)d_1\right)$$

$$= (A_1 \cap C_1)\left((d_1^{\,-1} d_1)\,(c_1\, d_1)\right)$$

$$= (A_1 \cap C_1)(c_1\, d_1)$$

Hence $(A_1 \cap C_1) \in F_1(G_1)$ is normal.

Similarly,

$$(A_1 \cup C_1)^{\phi} = (A_1 \cup C_1)$$

$$(A_1 \cup C_1)(c_1\, d_1) = (A_1 \cup C_1)^{\phi}(c_1\, d_1)$$

$$= (A_1 \cup C_1)(c_1\, d_1)^{\phi}$$

$$= (A_1 \cup C_1)\left(d_1^{\,-1}(c_1\, d_1)d_1\right)$$

$$= (A_1 \cup C_1)\left(d_1^{\,-1} d_1)\,(c_1\, d_1)\right)$$

$$= (A_1 \cup C_1)(c_1\, d_1)$$

Hence $(A_1 \cup C_1) \in F_1(G_1)$ is also normal.

References

[1] A.Rosenfeld.: Fuzzy group. J.Math.Anal.Appl.35,512-517 (1971).

[2] S.Sebastian and S.Babunder. Fuzzy groups and group homomorphism.Fuzzy sets and systems.8,397-401 (1996).

[3] S. Abou-zaid, On fuzzy subgroup, Fuzzy sets and systems,55. 1993, pp. 237–240.

[4] N.Ajmal and A.S.Prajapati,Fuzzy cosets and fuzzy normal subgroup, Inform.Sci, 64 (1992) 17-25.

[5] R.Biswas,Fuzzy subgroup and anti fuzzy subgroup.Fuzzy sets and systems.35,121-124 (1990).

[6] D.S.Malik,and J.N.Mordeson,Fuzzy subgroup and abelian group.Chinese.J.Math(Taipei).19,129-145 (1991).

[7] J.M.Anthony,and H.Sherwood,Fuzzy subgroup redefined,J.Math.Anal.Appl.69,124-130 (1979).

[8] J.M.Anthony,and H.Sherwood,A characterization of fuzzy subgroup, J.Math.Anal.Appl,69 (1979) 297-305.

[9] P.S.Das,Fuzzy groups and level subgroups J.Math.Anal.Appl.,84 (1981) 264-269.

[10] V.N.Dixit,R.kumar,and N.Ajmal,Level subgroups and union of fuzzy subgroups. Fuzzy Sets and Systems, **37** (1990) 359-371. and Technology, 1994, Vol. 2, pp. 87-98.

[11] M.S.Eroglu, The homomorphic image of a fuzzy subgroup is always a fuzzy subgroup, Fuzzy Sets and Systems, **33** (1989) 255-256.

[12] I.J.Kumar,P.K.Saxena,and P.Yadav,Fuzzy normal subgroups and fuzzy quotients, Fuzzy sets and systems, 46 (1992) 121-132.

[13] N.P.Mukherjee,and P.Bhattacharya,Fuzzy normal subgroups and fuzzy cosets: Information Science, 34 (1984) 225-239.

[14] M.T.A.Osman, On the direct product of fuzzy subgroups, Fuzzy sets and sysems, 12 (1984) 87-91.

[15] L.A.Zadeh. Fuzzy sets, Inform.Control, **8** (1965) 338-353.

[16] S.M.A.Zaidi, and Q.A.Ansari,: Some results of categories of L-fuzzy subgroups.Fuzzy sets and systems, **64** (1994) 249-256. Information Technology, 2012, Vol. 2, pp. 527-531. automata. Soft Computing, 2013 (Communicated).

[17] Y.Yu, A theory of isomorphism of fuzzy groups, Fuzzy system and Math.2,(1988),57-68.

[18] Wu W (1981) Normal fuzzy subgroups. Fuzzy Math. 1: 21-30

[19] Ajmal N, Thomas K V (1994) The lattice of fuzzy subgroups and fuzzy normal subgroups. Inform. Sci. 76: 1-11

[20] Ajmal N (1995) The lattice of fuzzy normal subgroups is modular. Inform. Sci. 83: 199-209 18.