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QUALITATIVE ASPECTS OF FLOW OF GROUND-WATER IN HETEROGENEOUS POROUS SUBSTRATE ON A SLOPPING BEDROCK

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Abstract: In this paper, we have mathematically analyzed the problem of groundwater flow given by Pavlovsky [1] by the method of two parameter singular perturbation with variable coefficient. The differential system governing the groundwater flow, yields a second order linear differential equation in which the coefficient of first and second order derivative consists of small parameter together with permeability factor. The constant term in the equation also consists a permeability factor. To analyze the problem we have, here, considered the permeability of the soil as a function of both time and special coordinate. Here, in this paper, we have discussed the problem for the flow of groundwater flow in heterogeneous porous media on a sloping bedrock**.**

***Index Terms* - heterogeneous porous, ground water flow, porous media, singular perturbation.**

I. INTRODUCTION

In this paper, we have mathematically analyzed the problem of groundwater flow by the method of two parameter singular perturbation with variable coefficient. The differential system governing the groundwater flow, yields a second order linear differential equation in which the coefficient of first and second order derivative consists of small parameter together with permeability factor. The constant term in the equation also consists a permeability factor. To analyze the problem we have, here, considered the permeability of the soil as a function of both time and special coordinate. Here, in this paper, we have discussed the problem for the flow of groundwater flow in heterogeneous porous media on a sloping bedrock.

The seepage problem of groundwater in homogeneous soil on slightly inclined bedrock has been discussed by Pavlovsky [1]. He has discussed four different cases, two of these the flow proceeds down the bedrock slope and in the other two it proceeds up the slope. In this work, we consider the seepage of groundwater down a slopping bedrock in soil which is heterogeneous in vertical direction. Water, from the head reservoir, flows into adjacent heterogeneous soil standing on the inclined bedrock and after seeping over considerable distance falls into a trail reservoir. Verma [2] has found that free surface, for seepage in the heterogeneous soil on an inclined bedrock is represented by an arc of rectangular hyperbola which is contrary to that of free surface found by Pavlovsky for homogeneous soil. Verma [5] has proved that the free surface for seepage in a two layered soil with an inclined boundary is a falling surface represented by an arc of rectangular whose concavity is downwards. But, because of our particular interest in analytical results, we have applied a two-parameter singular perturbation method with variable coefficients. We are particularly interested in determining the analytical solution by applying the two-parameter singular perturbation method, its convergence and the existence of the solution.

II. STATEMENT OF THE PROBLEM

Water from the head reservoir flows into adjacent soil which stands on an inclined bedrock and exhibits the heterogeneity in the vertical direction. After seeping over considerable distance it falls into tail reservoir. Choose a horizontal line at the bottom of the tail reservoir as the x-axis, a vertical line beside it as z-axis. The inclined boundary is the line $z = -mX$, where m = tanα is the slope.

III. MATHEMATICAL MODELLING AND GOVERNING EQUATIONS

The seepage velocity u is given by Darcy's law as

$$u = -K(z)\frac{\partial h}{\partial X} \qquad (3.1)$$

Where h is the piezometric (hydraulic) head and K(z) the seepage coefficient of the porous medium which varies with z. Since the flow of groundwater takes place over considerable distance, the analysis may be based on hydraulic theory. In hydraulic theory **[3]** the piezometric head 'h' is equal to height of the free surface (if we neglect the atmospheric pressure) and the flow elements depends on X alone. The flow rate q_x is given by

$$q_x = -\int_0^h K(z)\frac{\partial h}{\partial X}dz \qquad (3.2)$$

Where z=0 is the foot and z=h, the top of vertical section at a distance X for the q_x is measured.

The equation of continuity is for the phenomena is given by,

$$\frac{dq_x}{dX} = 0 \qquad (3.3)$$

From equation (3.3) we have,

$$q_x = \text{Constant} = q \qquad (3.4)$$

By the work of Polubarinova-Kochina [3] for definiteness, the seepage coefficient of flow region is given by a continuous linear relationship of the form $K(z) = K_0(1 - \lambda z)$ where K_0 and λ are some real constants, then equations (3.2) and (3.4) give

$$q = -K_0\frac{dh}{dX}\int_{-mX}^{h}(1-\lambda z)dz \qquad (3.5)$$

Since $\frac{dh}{dX}$ is independent of z, performing the integration, we get

$$q = -K_0\frac{dh}{dX}|(h+mX) - (\lambda/2)(h^2 - m^2X^2)| \qquad (3.6)$$

So that

$$\frac{dX}{dh} = P(h) + QX + RX^2 \qquad (3.7)$$

$$\text{where, } P = -\frac{k_0}{q}|h - (\lambda/2)h^2| \qquad (3.8)$$

$$Q = -\frac{k_0 m}{q} \qquad (3.9)$$

$$R = -\frac{k_0\lambda m^2}{2q} \qquad (3.10)$$

Equation (3.7) is the generalized Riccati's equation. To solve it let

$$X = -\frac{1}{R}\frac{1}{t}\frac{dt}{dh} \qquad (3.11)$$

Using (3.11), equation (3.7) reduces to

$$\frac{d^2t}{dh^2} - Q\frac{dt}{dh} + PRt = 0 \qquad (3.12)$$

This is a second order differential equation whose solution has been found by applying a two-parameter singular perturbation technique [4].

We associate an appropriate initial and boundary condition to problem (3.12) as

$$\left|\begin{array}{ll} t(h_0) = c & at\ X = 0\ (i.e., h = h_0) \\ t(h_L) = d & at\ X = X_L(i.e., h = h_L) \end{array}\right. \qquad (3.13)$$

In the equation (3.12), Verma [2] has considered Q and R as constants. Here, we have considered permeability 'K_0' as varying factor of soil and therefore, P, Q and R will be variable, in particular P, Q, R will be a function of special coordinate and time.

Setting $x = \frac{h-h_0}{h_L-h_0}$; $f = \frac{t}{(d-c)(h_L-h_0)^2}$ (3.14)

The problem (3.12-3.13) reduced to

$$\varepsilon \frac{d^2 f}{dx^2 + a(x)} \mu \frac{df}{dx} - b(x) f = 0 \qquad (3.15)$$

$$f(0) = \alpha, \qquad f(1) = \beta \qquad (3.16)$$

Where $\varepsilon = q^2,\ a(x) = k_0(x,t),\ \mu = mqH,\ h = h_L - h_0$

And $b(x) = \frac{k_0(x,t)\lambda m^2}{2} (H_x + h_0) \left|\frac{\lambda}{2}(H_x + h_0) - 1\right|$ For all $x \in [0,1], a(x) > 0, b(x) > 0$.

Now the flow rate 'q' in the porous medium is very-very small and m=tanα, the slope of the bedrock is also sufficient small. Therefore, we consider ε and μ, defined above, as a perturbation parameter and hence keeping in mind the definition of a(x) and b(x), we apply the technique of two parameter singular perturbation method with variable coefficient.

IV. Analytic Solution of The Problem

Here, we have considered $(\varepsilon/\mu^2) \to 0$ as $\mu \to 0$

The auxiliary polynomial equation for (3.15) is

$$\varepsilon D^2 + \mu\, a(x) D - b(x) = 0 \qquad (4.1)$$

With roots.

$$D_1(x,\mu,\varepsilon) = \frac{1}{\mu}\left|\frac{b(x)}{a(x)} - \frac{\varepsilon}{\mu^2}\frac{b^2(x)}{a^2(x)} + \cdots\right| = \frac{1}{\mu} d_1(x, \varepsilon/\mu^2)$$

$$D_2(x,\mu,\varepsilon) = -(\mu/\varepsilon)\left|a(x) + \frac{\varepsilon}{\mu^2}\frac{b(x)}{a(x)} - \frac{\varepsilon^2}{\mu^4}\frac{b^2(x)}{a^3(x)} + \cdots\right| = -(\mu/\varepsilon)\, d_2(x, \varepsilon/\mu^2)$$

Now writing

$$d_1\left(x, {}^{\varepsilon}/_{\mu^2}\right) = \sum_{j=0}^{\infty} d_{1j}(x)(\varepsilon/\mu^2)^j$$

$$d_2\left(x, {}^{\varepsilon}/_{\mu^2}\right) = \sum_{j=0}^{\infty} d_{2j}(x)(\varepsilon/\mu^2)^j$$

Here both d_1 and d_2 are positive for (ε/μ^2) sufficiently small. Since the root $\frac{1}{\mu}d_1$ of (4.1) tends to ∞ as μ→0, we associate with it, a boundary layer at x=1. While we associate a boundary layer at x=0 with root $(\mu/\varepsilon)d_2$ which tends to – ∞ as μ→0.

Thus, we set

$$f(x) = exp.\left|\frac{1}{\mu}\int_1^X d_1(s,\varepsilon/\mu^2)\, ds\ \ \alpha(x,\mu)\right| + exp.\left|-(\mu/\varepsilon)\int_0^x d_2(s,\varepsilon/\mu^2)\, ds\ \ \beta(x,\mu)\right| \qquad (4.2)$$

Substituting (4.4.2) in (4.3.15) we get,

$$\left|\varepsilon\alpha'' + \frac{2\varepsilon}{\mu} d_1\alpha' + \frac{\varepsilon}{\mu} d_1'\alpha + \mu a(x)\alpha' + \left(\frac{\varepsilon}{\mu^2} d_1^2 + a(x) d_1 - b(x)\right)\alpha\right| \text{X } exp.\left|\frac{1}{\mu}\int_1^x d_1(s,\varepsilon/\mu^2) ds\right|$$

$$+\left|\varepsilon\beta'' - 2\mu d_2\beta' - \mu d_2'\beta + \mu\, a(x)\beta'^{+\left(\frac{\mu^2}{\varepsilon}d_2^2 - \frac{\mu^2}{\varepsilon}a(x)d_2 - b(x)\right)\beta}\right| \text{ X } exp.\left|-\frac{\mu}{\varepsilon}\int_0^x d_2(s,\varepsilon/\mu^2) ds\right| = 0 \qquad (4.3)$$

Keeping in mind the definition of d_1 and d_2 we take

$$\frac{\varepsilon}{\mu}\alpha'' + \frac{2\epsilon}{\mu^2} d_1\alpha' + \frac{\varepsilon}{\mu^2} d_1'\alpha + a(x)\alpha' = 0 \qquad (4.4)$$

$$\frac{\varepsilon}{\mu}\beta'' - 2d_2\beta' + d_2'\beta + a(x)\beta' = 0 \qquad (4.5)$$

Where we let,

$$\alpha(x,\mu) = \sum_{m,n=0}^{\infty} a_{mn}(x)(\varepsilon/\mu^2)^m (\varepsilon/\mu^2)^n = A(x, \varepsilon/\mu, \varepsilon/\mu^2) \qquad (4.6)$$

$$\beta(x,\mu) = \sum_{m,n=0}^{\infty} b_{mn}(x)(\varepsilon/\mu)^m (\varepsilon/\mu^2)^n = B(x, \varepsilon/\mu, \varepsilon/\mu^2) \qquad (4.7)$$

Substituting these results in (4.4) and (4.5) and then equating the like powers of the small parameter (ε/μ) and (ε/μ^2), we get

$$a(x)a''_{m-1,n} - 2\sum_{k=0}^{n-1} a'_{m,k}\, d_{1_{n-k-1}} - \sum_{k=0}^{n-1} a_{m,k} d'_{1_{n-k-1}}$$

$$(a(x)b_{mn})' = b''_{m-1,m} - 2\sum_{k=0}^{n-1}(b_m, k d_{2_{n-k}})' + \sum_{k=0}^{n-1} b_{m,k} d'_{2_{n-k}}$$

Thus, each a_{mn} and each $a(x)b_{mn}$ can be successively determined within the additive constants a_{mn} and b_{mn}. In particular

$$a_{00}(x) = \lambda_{00}$$

$$a_{01}(x) = -\lambda_{00}\int_1^x \frac{1}{a(s)}\left(\frac{a(s)}{a(b)}\right)' ds + \lambda_{01}$$

$$a_{10}(x) = \lambda_{10}$$

$$b_{00}(x) = \frac{\sigma_{00}}{a(x)}$$

$$b_{01}(x) = \frac{1}{a(x)}\left|-2\sigma_{00}\frac{b(x)}{a^2(x)} + \sigma_{00}\int_0^x \frac{1}{a(s)}\left(\frac{b(s)}{a(s)}\right)' ds + \sigma_{01}\right.$$

$$b_{10}(x) = \frac{1}{a(x)}\left|\sigma_{00}\left(\frac{1}{a(x)}\right)' + \sigma_{10}\right|$$

Now

$$f(0) = A\left(0, {}^{\varepsilon}/_{\mu}, {}^{\varepsilon}/_{\mu^2}\right)\ exp.\left|\frac{1}{\mu}\int_1^0 d_1\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right|$$

$$+\sum_{m,n=0}^{\infty} b_{mn}(0)\left({}^{\varepsilon}/_{\mu}\right)^m\left({}^{\varepsilon}/_{\mu^2}\right)^n$$

Since, $A\left(10, {}^{\epsilon}/_{\mu}, {}^{\varepsilon}/_{\mu^2}\right) exp.\left|\frac{1}{\mu}\int_1^0 d_1\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right|$ is exponentially small, we select

$$\sigma_{00} = a(0)f(0)$$

$$\sigma_{mn} = -b'_{m-1,n}(0)$$

Which implies that $b_{mn}(0)=0$

Similarly

$$f(1) = \sum_{m,m=0}^{\infty}\lambda_{mn}\left({}^{\varepsilon}/_{\mu}\right)^m\left({}^{\varepsilon}/_{\mu^2}\right)^n + B\left(1, {}^{\varepsilon}/_{\mu}, {}^{\varepsilon}/_{\mu^2}\right) exp.\left|-\frac{\mu}{\varepsilon}\int_0^1 d_2\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right|$$

We choose, $f(1)=\lambda_{00}$; $\lambda_{mn}=0$, otherwise using the above results in the equation (4.2), we get

$$f(x) = f(1)\left[1 - \frac{\varepsilon}{\mu^2}\int_1^x \frac{1}{a(s)}\left[\frac{b(s)}{a(s)}\right]' + \frac{\varepsilon}{\mu}0 + \cdots\right] X\ exp.\left[\frac{1}{\mu}\int_1^x d_1\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right]$$

$$+\frac{a(0)f(0)}{a(x)}[1 + {}^{\varepsilon}/_{\mu^2}\left(-2\frac{b(x)}{a^2(x)} + \frac{2b(0)}{a^2(0)}\right) + \int_0^x \frac{1}{a(s)}\left[\frac{b(s)}{a(s)}\right]' ds) + \frac{\varepsilon}{\mu}\left(\frac{1}{a(x)}\right)' - \left(\frac{1}{a(x)}\right)' + \cdots]X\ exp.\left[-\frac{\mu}{\varepsilon}\int_0^x d_2\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right] \quad (4.8)$$

Where ε, μ, a(s) and b(s) are the known functions.

V. Uniqueness And Existence Of Solution:

In this section, we prove the existence of the boundary value problem (3.15-16) has a unique solution f(x).

Consider $a_{mn}(x)$ and $b_{mn}(x)$ defined as, above in (4.6-4.7).

We set

$$f(x) = \left|\sum_{\substack{m,n\geq 0\\ m+n\leq N}} a_{mn}(x)\left({}^{\varepsilon}/_{\mu}\right)^m\left({}^{\varepsilon}/_{\mu^2}\right)^n\right| exp.\left|\int_1^x d_1\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right|$$

$$+\left|\sum_{\substack{m,n\geq 0\\ m+n\leq N}} b_{mn}(x)\left({}^{\varepsilon}/_{\mu}\right)^m\left({}^{\varepsilon}/_{\mu^2}\right)^n\right| exp.\left|-\frac{\mu}{\varepsilon}\int_0^x d_2\left(s, {}^{\varepsilon}/_{\mu^2}\right) ds\right| + \left({}^{\varepsilon}/_{\mu^2}\right)^{N+1} T_N(X,\mu)$$

Where $T_n = 0(\mu)$ for $x \in [0,1]$

Introduce $z(x,\mu) = \mu a(x)f' - b(x)f$ then equation (4.6) implies that

$$z(t) = z(0)exp.\left|-\frac{\mu}{\varepsilon}\int_0^t a(s)h(s,\mu,\varepsilon)ds\right| - \frac{1}{\mu}\int_0^t f(s)\theta(s,t,\varepsilon,\mu)ds$$

Where $h(s,\mu,\varepsilon) = 1 + \frac{\varepsilon}{\mu^2 a^2(s)}|b(s) - \mu a'(s)|$

And

$$\theta(s,t,\varepsilon,\mu) = \left|\frac{b^2(s) - \mu a'(s)b(s) + \mu a(s)b'(s)}{a(s)}\right| exp.\left|-\frac{\mu}{\varepsilon}\int_0^t a(r)h(r,\mu,\varepsilon)dr\right|$$

Then

$$f(x) = (1)\left|exp.\left(\frac{1}{\mu}\int_1^x \frac{b(s)}{a(s)}ds\right)\right| - \frac{z(0)}{\mu}\int_x^1\left|\frac{1}{a(t)}\ exp.\left(\frac{1}{\mu}\int_t^x \frac{b(s)}{a(s)}ds\right) X\ exp. -\frac{\mu}{\varepsilon}\int_0^t a(r)h(r,\mu,\varepsilon)dr\right| dt$$

$$+\frac{1}{\mu^2}\int_x^1 \left|\frac{1}{a(t)}\, exp.\left(\frac{1}{\mu}\int_t^x \frac{b(s)}{a(s)}ds\right) X \int_0^t f(s)\theta(s,t,\varepsilon,\mu)ds\right| dt$$

Which, by evaluating at x=0 implies that

$$-\frac{z(0)}{\mu}\int_0^1 \left|\frac{1}{a(t)} exp.\right|\left|\left(-\frac{\mu}{\varepsilon}\int_0^t \left|a(r)h(r,\mu,\varepsilon)+\frac{\varepsilon}{\mu^2}\frac{b(r)}{a^2(r)}\right|dr\right)\right| dt$$

$$= f(0) - f(1)\, exp.\left|\frac{1}{\mu}\int_1^0 \frac{b(s)}{a(s)}ds\right|$$

$$-\frac{1}{\mu^2}\int_0^1 \left|\frac{1}{a(t)}\, exp.\left(\frac{1}{\mu}\int_t^0 \frac{b(s)}{a(s)}ds\right)\int_0^t f(s)\theta(s,t,\mu,\varepsilon)ds\right| dt$$

Now we introduce

$$\theta(x,\varepsilon,\mu) = \frac{\int_x^1 \left|\frac{1}{a(t)}\, exp.\left(\frac{1}{\mu}\int_t^x \frac{b(s)}{a(s)}ds\right) exp.\left(-\frac{\mu}{\varepsilon}\int_0^t a(r)h(r,\mu,\varepsilon)\right)dr\right|}{\int_0^1 \left|\frac{1}{a(t)}\, exp.\left(-\frac{\mu}{\varepsilon}\int_0^t a(r)h(r,\mu,\varepsilon)+\frac{\varepsilon}{\mu^2}\frac{b(r)}{a(r)}dr\right| dt}$$

And

$$f_0(x) = f(1)exp.\left(\frac{1}{\mu}\int_1^x \frac{b(s)}{a(s)}ds\right) + Q(x,\varepsilon,\mu)\left|y(0) - y(1)exp.\left(\frac{1}{\mu}\int_1^0 \frac{b(s)}{a(s)}ds\right)\right|$$

Hence, we have

$$f(x) = f_0(x) + \frac{1}{\mu^2}\left|\int_x^1 exp.\left(\frac{1}{\mu}\int_t^x \frac{b(s)}{a(s)}ds\right) - Q(x,\varepsilon,\mu)\int_0^1 exp.\left(\frac{1}{\mu}\int_t^0 \frac{b(s)}{a(s)}ds\right)\right| X \left|\frac{1}{a(t)}\int_0^t f(s)\theta(s,t,\varepsilon,\mu)ds.\,dt\right| \qquad (5.1)$$

Equation (5.1) can be solved by successive approximation, we define

$$f_j(x) = f_0(x) + \frac{1}{\mu^2}\left|\int_x^1 exp.\left(\frac{1}{\mu}\int_t^x \frac{b(s)}{a(s)}ds\right) - Q(x,\varepsilon,\mu)\int_0^1 exp.\left(\frac{1}{\mu}\int_t^0 \frac{b(s)}{a(s)}ds\right)\right| X \left|\frac{1}{a(t)}\int_0^t f_{j-1}(x)\theta(s,t,\varepsilon,\mu)ds.\,dt\right| \qquad (5.2)$$

Interchanging the order, the integration in (5.2) and taking estimate, we can prove that,

$$\|f_j(x) - f_{j-1}(x)\| \le \frac{M\varepsilon}{\mu^2}\|f_{j-1}(x) - f_{j-2}(x)\| \qquad (5.3)$$

Where M is positive constant and ||.|| is supermom norm on [0,1]. Inequality (5.3) implies that the integral operator in (5.1) is contractive for (ε/μ^2) sufficiently small. Thus $f(x) = \lim_{j\to\infty} f_j(x)$ exists for all $x \in [0,1]$ for μ sufficiently small and is unique solution of the boundary value problem.

VI. Asymptotic convergence of The Solution

Using the definitions of $a_{mn}(x)$ and $b_{mn}(x)$, as defined in section (4.4), we have

$$\varepsilon T_N'' + \mu\, a(x)\, T_N' - b(x)T_N = \mu C_N(x,\mu) \qquad (6.1)$$

Where $C_N(x,\mu)$ is bounded. Moreover $T_N(0,\mu)$ and $T_N(1,\mu)$ are exponentially very small. The boundedness (T_N/μ) follows by the maximum-minimum principal argument.

Suppose T_N has a non-zero maximum at $x \in [0,1]$ then $T_N'(\xi) = 0$ and $T_N''(\xi) < 0$

Now

$$b(\xi)\, T_N(\xi,\mu) \le -\varepsilon\, T_N''(\xi,\mu) - \mu\, a(\xi)\, T_N'(\xi,\mu) + b(\xi)T_N(\xi,\mu) \le \mu C_U \text{ Where } C_U = \max_{x\varepsilon|0,1|} C_N(x,\mu)$$

Hence, $T_N(\xi,\mu) \le \frac{\mu C_U}{b_L}$, Where $b_L = \min_{x\varepsilon|0,1|} b(x) > 0$

Similarly, if T_N has negative minimum at $x \in [0,1]$ then

$$T_N'(\xi) = 0 \;; \quad T_N''(\xi) > 0 \implies -b(\xi)T_N(\xi,\mu) \le \varepsilon T_N''(\xi,\mu) + \mu\, a(\xi)T_N'(\xi,\mu) - b(\xi)T_N(\xi,\mu) \le \mu\, C_U$$

Therefore,

$$T_N(\xi,\mu) \ge -\frac{\mu C_U}{b_L}$$

Since T_N is exponentially small at x=0 and x=1, we have

$$|T_N(x,\mu)| \leq \mu \frac{C_U}{b_L} \text{ for all } x \in [0,1]$$

Hence solution converges to a finite limit.

VII. Conclusion

Verma [6] has proved that the free surface for seepage in a two layered soil with an inclined boundary is a falling surface represented by an arc of rectangular whose concavity is downwards. But, because of our particular interest in analytical results, we have applied a two-parameter singular perturbation method with variable coefficient. For definiteness we have assumed that the permeability of both the layered is a function of (x, t), i.e., permeability of the layered vary at each and every point of the media as time varies. We have discussed the solution of the problem and uniqueness and existence together with asymptotic convergence of the obtained general solution. Also, we have showed here that when the slope of the inclined boundary is small and the flow rate is sufficiently small, the free surface of water falls partly represented by St. line and partly represented by an arc of negative exponential curve

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