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STOCHASTIC COMPARISONS OF LIFETIME DISTRIBUTIONS USING *a.p.g.f* RATIOS

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Abstract:Discrete ageing classes can be classified according to various ageing concepts and stochastic orderings.This paper deals with stochastic comparisons of certain lifetime distributions with respect to their *a. p. g. f.s* discrete analogue of Laplace transforms. Based on ratios of a, p, g, f s, two types of orderings of lifetime distributions are developed and their properties are studied. Certain preservation properties of these ordering are also considered.

Index Terms - *a.p.g.f*, discrete ageing classes, ratio ordering, stochastic ordering.

1.Introduction

Even though discrete models are extensively used for studying the application of failure process that involves discrete trials, they deserve more attention. A large number of problems that arise with continuous time models are solved by using discrete time model. Most of the results derived for continuous random variables can be derived analogously for discrete variables. Literature related to discrete time models are comparatively less in number due to this very reason. Some authors have devoted to their study for developing the concept of discrete models and stochastic orders. For example, refer Asha et al (2016), Asha and Rejeesh (2007), Bracquemond and Gaudoin(2003), Bracquemond et al(2001), Goliforushano and Asadi (2008), Gupta et al. (1997), Gupta and Richard (1997), Jiang (2010), Kemp (2004), Lai (2013), Nanda and Sengupta (2005), Roy and Gupta (1999), Salvia and Bollinger (1982), Shaked et al. (1994, 1995), Xekalaki (1983), Xie et al.(2002), and the references therein.

This paper is organized into 3 sections. In section 2 we detail the preliminary concepts and definitions. $a.p.g.f.$ ratio ordering and its properties are discussed in section 3 followed by the conclusion.

2.Preliminary concepts and Definitions

In the study of various classes of lifetime distributions, stochastic ordering plays an important role. Various types of stochastic orderings are described in Shaked and Shanthikumar (1994, 2007), Szekli (1995) and Muller and Stoyan (2002). The vast majority of literature on the various criteria for ageing treat lifetime as continuous with only occasional references to the discrete case. Xekalaki (1983) points out that limitations of measuring devices and the fact that discrete models provide good approximations to their continuous counterparts necessitates assessment of reliability in discrete time.

The definitions and results relating to*IFR, IFRA, DMRL, NBU, NBUE, HNBUE, GHNBUE*, are given in Esary et al (1973), Golifourshni and Asadi (2008), Kemp (2004), Klefsjö (1983). In this section we present definitions, notation, and basic facts that are used throughout the paper.

2.1.Stochastic Orders

The objectives of a reliability study are understanding the failure phenomena, estimating and predicting reliability, optimization etc. In order to study the failure phenomena of the systems or units, we consider their failure time distribution. Since the failure time as well as the residual lifetime are random variables, several types of stochastic orders have been developed by various researchers (see Shaked and Shanthikumar (1994), Klefsjö (1983)).

i. Stochastic Order

Let X and Y be two random variables such that $P[X>u]\le P[Y>u]$ for all $u\in(-\infty,\infty)$. Then X is said to be smaller than Y in the usual stochastic order and is denoted by $X\le_{st} Y$.

ii. Hazard Rate Order

Let X and Y be two non-negative random variables with absolutely continuous distribution functions and hazard rate functions $r_1(\cdot)$ and $r_2(\cdot)$ respectively, such that $r_1(t)\ge r_2(t), t\ge 0$. Then X is said to be smaller than Y in the hazard rate order and is denoted by $X\le_{hr} Y$. It may be noted that if X and Y are two random variables such that $X\le_{hr} Y$, then $X\le_{st} Y$. It is clear that the hazard rate ordering would be appropriate for comparing the life lengths of identical devices, one operating in a more hazardous environment than the other, since the hazard rate can be interpreted as the probability of failure in the next instant of time given that the system has survived up to a specific time.

iii. Star Order

Suppose X has distribution function F and Y has distribution function G. Then X is said to be smaller than Y in star order, denoted by $X\le_{*} Y$, if $G^{-1}(F)$ is star shaped in x, that is, $\frac{G^{-1}(F(x))}{x}$ is increasing in $x\ge 0$.

iv. Supperadditive Order

Suppose that X has distribution function F and Y has distribution function G. Then, X is said to be smaller than Y in the supperadditive order if $G^{-1}(F)=\bar{G}^{-1}(\bar{F})$ is supperadditive in x, that is, $G^{-1}(F(x+y))\ge G^{-1}(F(x))+G^{-1}(F(y))$, $x,y\ge 0$.

v. Convex Order

Let X and Y be two random variables such that $E(\phi(X))\le E(\phi(Y))$ for all convex functions $\phi: R\to R$, provided the expectations exist. Then X is said to be smaller than Y in the convex order and is denoted by $X\le_{cx} Y$

vi. Laplace Transform Order

Let X and Y be two non-negative random variables such that,

$E(e^{-sX})\le E(e^{-sY})$ for all $s>0$. (2.1)

Then X is said to be smaller than Y in the Laplace transform order and is denoted by $X\le_{L} Y$. The above ordering concepts among probability distributions based on comparison of their Laplace transforms are taken from Klefsj o (1983). It differs from the definition used by Stoyan (1983), where the reversed inequality is used in (2.1). In the sequel of our work we use the definition given by Klefsjö (1983).

2.2. Discrete Ageing Classes

Accurate distribution of the life of a component or a system is usually not available in practical situations. Ageing properties play an important role in modelling ageing or wear out process. The notion of ageing was first introduced in Barlow and Marshall (1964). Prior to that the physics of failure mainly focused on the properties of specially chosen families of failure distributions. Barlow and Proschan (1975) generalized the main assumptions of the theory of ageing distributions. Important contributions to this theory were made by Bryson and Siddiqui (1969), Proschan and Hollander (1984), Cox and Oakes etc.Recently there is increasing interest for reliability modeling and analysis in the discrete time domain. Let X denote a discrete lifetime random variable whose probability mass function and cumulative distribution function are given by, $p_k=P[X=k], k=0,1,2,\ldots$ and $P_k=P[X\le k]$.

Then the failure rate function is defined as $h_k = \frac{p_k}{\bar{P}_{k-1}}$, where $\bar{P}_k = P[X > k]$. It gives the conditional probability of the failure of the device at time k, given that it has not failed by time $k-1$. The failure rate function uniquely determines the distribution. Shaked and Shanthikumar (1994) proved the following necessary and sufficient conditions for a sequence $\{h_k, k \geq 1\}$ to be a failure rate function.

1. For all $k < m,\ h_k < 1$ and $h_m = 1$, the distribution is defined over $\{1,2,\ldots,m.\}$
2. For all $k \in N^+ = \{1,2,\ldots\}, 0 \leq h_k \leq 1$ and $\sum_{i=1}^{\infty} h_i = \infty$. The distribution is defined over $k \in N^+$.The mean residual life (or mean remaining life) at time k is defined as $\mu_k = \sum_{n=k+1}^{\infty} \frac{\bar{P}_n}{\bar{P}_k}$. The mean lifetime $\mu = E[X] = \sum_{k=0}^{\infty} \bar{P}_k < \infty$.

Now we shall consider the definitions of discrete version of the above mentioned ageing classes for completeness. For further properties see Esary et al (1973), Klefsjö (1982).

Definition 2.1 *A discrete survival probability* $\bar{P}_k = \sum_{j=k+1}^{\infty} p_j$, *with support on* $\{0,1,2,\ldots,\}$ *probability mass function* $p_k = \bar{P}_{k-1} - \bar{P}_k$ *for* $k = 1,2,3,\ldots$ *and* $p_0 = 1 - \bar{P}_0$ *is said to be*

1. IFR if $\frac{\bar{P}_{k+1}}{\bar{P}_k}$ is decreasing in k .For IFR distributions, Salvia and Bollinger (1982) shows that $\bar{P}_{k-1} \leq (1-h_0)^k \approx \exp(-h_0^k)$ and that $\mu \leq \frac{(1-h_0)}{h_0}$.

2. IFRA if$(\overline{P_k})^{-1/k}$is decreasing in k .The class of distributions distinguished by $IFRA$ property was introduced for continuous random variables by Birnbaum et al (1966) in an attempt to find a new class of life distributions that reflect the phenomenon of wear-out. Klefsj$\ddot{o}$(1982) has considered the discrete $IFRA$ class, preferring to define it in terms of the behavior of $[\bar{F}(x)]^{1/x}$ where $\bar{F}(x) = P[X > x]$, as in the continuous case.
3. DMRL if $\sum_{j=k}^{\infty} \frac{\bar{P}_j}{\bar{P}_k}$ is decreasing in k .
4. NBU if $\bar{P}_{k+j} \leq \bar{P}_k \bar{P}_j$.

The quantity $\frac{\bar{P}_{k+j}}{\bar{P}_k}$ represents the survival function of a unit of age k or the conditional probability that a unit of age k will survive for an additional j units of time. At $k = 0, \frac{\bar{P}_{k+j}}{\bar{P}_k} = \bar{P}_j$ is the survival function of a new unit and accordingly the ageing of the device can be studied by comparing $\frac{\bar{P}_{k+j}}{\bar{P}_k}$ and $\bar{P}_j$. Thus $\bar{P}_{k+j} \leq \bar{P}_k \bar{P}_j$ if and only if the older system that has aged has no better chance of surviving for a duration of j than does a new system. In other words, the new unit is better than the used one or NBU.

5. NBUE if $\sum_{j=k}^{\infty} \bar{P}_j \leq \bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j$. This condition states that the expected remaining life of a unit surviving age k is not larger than the expected life of a new unit so that the new unit is better than the unit of age k in terms of the expected life length.
6. HNBUE if $\mu = \sum_{j=0}^{\infty} \bar{P}_j$ is finite and $\sum_{j=k}^{\infty} \bar{P}_j \leq \mu(1-1/\mu)^k,\ \ k = 0,1,2,\ldots$.

Corresponding dual classes of distributions are also defined by reversing the direction of monotonicity or inequality in the above definitions.

Let f denote the probability mass function ($p.m.f.$) of a non-negative integer-valued random variable X with probability generating function $(p.g.f.)\, g(\cdot)$ defined by

$$g(s) = E(s^X), \quad 0 < s \leq 1.$$

Then the $a.p.g.f.\, G(\cdot)$ of f is defined as

$$G(s) = E(1-s)^X, \quad 0 < s \leq 1. \qquad (2.2)$$

As a discrete analogue of Laplace transform ordering, Jayamol and Jose (2008) introduced $a.p.g.f.$ ordering as follows.

Definition 2.2 *Suppose that* X *and* Y *are two non-negative integer-valued random variables with* $p.m.f.s$ f_1 *and* f_2 *and* $a.p.g.f.s$ $G_1(s)$ *and* $G_2(s)$ *respectively. Then* X *is said to be smaller than* Y *(or equivalently,* f_1 *is smaller than* f_2 *) in* $a.p.g.f.$ *ordering if* $G_1(s) \le G_2(s)$, *for* $0 \le s \le 1$. *It is denoted by* $X \le_G Y$ *(or equivalently, write* $f_1 \le_G f_2$*).*

Result 2.1 *Let* X *be a non-negative integer-valued random variable, that possess* r^{th} *moments* μ_r $r=1,2,\ldots$ *Then* $G(s) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu^r$.

For Properties of $a.p.g.f$ ordering refer Jayamol and Jose (2008, 2020)

3. *a.p.g.f.* *Ratio Ordering and its Properties*

In this section we study two notions of stochastic comparisons of non-negative integer-valued random variables using the ratios of their $a.p.g.f.s$. Various properties of this orderings are considered.

Let X be a non-negative integer-valued random variable, with distribution function $P_k = P[X \le k],\ k = 0,1,2,\ldots$ and $\bar{P}_k = 1 - P_k$ be the survival function. It can be proved that $a.p.g.f.$ of $\bar{P}_k$,

$$G^*(s) = \frac{1-G(s)}{s}, 0 < s < 1.$$

Thus $G(s) = 1 - sG^*(s).(3.1)$

Here we are introducing two orders based on ratios of $a.p.g.f.s$ as a discrete analogue of Laplace transform ratio ordering introduced by Shaked and Wong (1997).

Definition 3.1 *Consider two non-negative integer-valued random variables* X *and* Y *with* $a.p.g.f.s\, G_X(s)$ *and* $G_Y(s)$ *respectively. Also let* $a.p.g.f.s$ *s of their respective survival functions be* $G_X^*(s)$ and $G_Y^*(s)$ *. Then*

1. X is said to be smaller than Y in $a.p.g.f.$ ratio ordering (denoted by $X \le_{G-r} Y$) if
$$\frac{G_Y(s)}{G_X(s)} \text{ is decreasing in } 0 < s \le 1.$$
2. X is said to be smaller than Y in reverse $a.p.g.f.$ ratio ordering (denoted by $X \le_{r-G-r} Y$) if
$$\frac{1-G_Y(s)}{1-G_X(s)} \text{ is decreasing in } 0 < s \le 1.$$

Using equation (3.1), Definition (3.1) can be equivalently written as,

Definition 3.2 *Consider two non-negative integer-valued random variables* X *and* Y *with* $a.p.g.f.s\, G_X(s)$ *and* $G_Y(s)$ *respectively. Also let* $a.p.g.f.s$ *s of their respective survival functions be* $G_X^*(s)$ and $G_Y^*(s)$ *. Then*

1. X is said to be smaller than Y in $a.p.g.f.$ ratio ordering (denoted by $X \le_{G-r} Y$) if
$$\frac{1-sG_Y^*(s)}{1-sG^*{}_X(s)} \text{ is decreasing in } 0 < s \le 1. \qquad (3.2)$$
2. X is said to be smaller than Y in reverse $a.p.g.f.$ ratio ordering (denoted by $X \le_{r-G-r} Y$) if
$$\frac{G_Y^*(s)}{G_X^*(s)} \text{ is decreasing in } 0 < s \le 1. \qquad (3.3)$$

These orderings have a variety of interpretations analogous to those in $\le_{r-Lt-r}$ ordering (Shaked and Wong (1997)), corresponding to models in reliability, insurance and maintenance. Some of them are

1. Suppose that a machine, with survival function $\bar{P}_k$, produces one unit of output per unit time when functioning. The present value of one unit produced at time k is $(1-s)^k$ where s is the discount rate. Then the expected present value of the total output produced during the lifetime of the machine is $G^*(s)$. Thus it can be seen that $X \leq_{r-G-r} Y$ implies the expected present value of a machine with lifetime Y, relative to the expected present value of a machine with lifetime X increases as s gets smaller.

2. $G^*(s)$ can be interpreted as the expected present value of the total cost that a person, whose remaining lifetime has the survival function $\bar{P}_k$, pays for an insurance policy where s is the discount rate.

3. Also $G^*(s)$ can be interpreted as expected present value of the total maintenance cost of a machine with survival function $\bar{P}_k$, where s is the discount rate.

Theorem 3.1 *Let X and Y be two non-negative integer-valued random variables that possess moments μ_r and λ_r respectively, $r = 0,1,2,\ldots$. Then $X \leq_{G-r} Y$ if and only if*

$$\frac{\sum_{r=0}^{\infty} \frac{t^r}{r!}\lambda_r}{\sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r} \text{ is decreasing in } 0 < s \leq 1, \text{ where } t = \log(1-s).$$

Proof

Proof follows from result (2.1) and from Definition 3.1(1).

Theorem 3.2 *Let X and Y be two non-negative integer-valued random variables that possess moments μ_r and λ_r respectively, $r = 0,1,2,\ldots$. Then $X \leq_{r-G-r} Y$ if and only if*

$$\frac{\sum_{r=1}^{\infty} \frac{t^r}{r!}\lambda_r}{\sum_{r=1}^{\infty} \frac{t^r}{r!}\mu_r} \text{ is decreasing in } 0 < s \leq 1, \text{ where } t = \log(1-s).$$

Proof

Proof follows from result (2.1) and Definition 3.1(2).

Definition 3.3 *Let X and Y be two non-negative integer-valued random variables. If $X \leq_{G-r} Y$ or $X \leq_{r-G-r} Y$, then $Y \leq_G X$ for $0 < s \leq 1$.*

Proof

We have $G_X(0) = G_Y(0) = 1$ and $G_X(1) = G_Y(1) = 0$.

$$\text{Thus, } X \leq_{G-r} Y \Leftrightarrow \frac{G_Y(s)}{G_X(s)} \leq \frac{G_Y(0)}{G_X(0)} = 1$$

$$\Leftrightarrow G_Y(s) \leq G_X(s)$$

$$\Leftrightarrow Y \leq_G X.$$

$$X \leq_{r-G-r} Y \Leftrightarrow \frac{1-G_Y(s)}{1-G_X(s)} \geq \frac{1-G_Y(1)}{1-G_X(1)} = 1$$

$$\Leftrightarrow 1-G_Y(s) \geq 1-G_X(s) \Leftrightarrow G_Y(s) \leq G_X(s)$$

$$\Leftrightarrow Y \leq_G X.$$

3.1 Preservation Property of $a.p.g.f.$ ratio ordering

The following theorem gives the preservation property of $a.p.g.f.$ ratio ordering under summation.

Theorem 3.3 *Let $n>0$ be a fixed integer. Let $X_1, X_2, \ldots, X_n$ be a set of independently distributed non-negative integer-valued random variables and let $Y_1, Y_2, \ldots, Y_n$ be another set of independently distributed non-negative integer-valued random variables. If $X_i \le_{G-r} Y_i, i=1,2,\ldots,n$. Then $\sum_{i=1}^{i=n} X_i \le_{G-r} \sum_{i=1}^{i=n} Y_i$*

Proof

We have the $a.p.g.f.\, G_{X_1+X_2+\cdots+X_n} = \prod_{i=1}^{n} G_{X_i}(s)$.

$$X_i \le_{G-r} Y_i \Leftrightarrow \frac{G_{Y_i}(s)}{G_{X_i}(s)} \text{ is decreasing in } s.$$

Thus $\dfrac{\prod_{i=1}^{n} G_{Y_i}(s)}{\prod_{i=1}^{n} G_{X_i}(s)} = \dfrac{G_{Y_1+Y_2+\cdots+Y_n}}{G_{X_1+X_2+\cdots+X_n}}$ is decreasing in *s*.

Let X and y be two independent random variables with respective distribution functions F and G and with $P(X>Y)>0$ and let $X_Y = [X-Y / X>Y]$ denote the discrete residual lifetime at random time or age . The survival function of X_Y is given by

$$\bar{F}_Y(i) = P[X-Y>i / X>Y] \tag{3.4}$$

$$= \frac{\sum_{y=0}^{\infty} \bar{F}(i+y)g(y)}{\sum_{y=0}^{\infty} \bar{F}(y)g(y)} \tag{3.5}$$

For more details refer (Stoyan (1983). For properties and details of the following definition refer Elbatal and Ahsanullah (2012) .

Definition 3.4 *A non negative random variable X is said to be discrete new better than used in probability generating function order (denoted by $F \in d-NBU_{pg}$ if and only if $X_t \le_{pg} X$ for all $t \in N$.*

Equivalently,

$$\sum_{x=0}^{\infty} s^x \bar{F}(x+t) \le F(t) \sum_{x=0}^{t} s^x \bar{F}(x), \quad for \ \ all \ \ t \in N, 0<s<1. \tag{3.6}$$

Theorem 3.4 *Let X and Y be two non negative integer valued random variables with survival functions $\bar{P}$ and $\bar{Q}$ respectively with $P(X>Y)>0$. Then $P \in d-NBU_{pg}$ if and only if $X <_G X_Y$.*

Proof

$p \in d-NBU_{pg}$ if and only if

$$\sum_{x=0}^{\infty} s^x \bar{P}(x+t) \le P(t) \sum_{x=0}^{t} s^x \bar{P}(x), \quad for \ \ all \ \ t \in N, 0<s<1.$$

By equation (3.6) $p \in d-NBU_{pg}$ if and only if

$$\sum_{x=0}^{\infty} s^x \bar{P}_Y(x) \le \sum_{x=0}^{\infty} s^x \bar{P}(x) \qquad (3.7)$$

Replacing 's' by '1-s' in equation (3.7) and from the following relation the result follows.

$$E(1-s)^X = 1 - s\sum_{k=0}^{\infty} \bar{P}_k (1-s)^k .$$

Conclusion

The notion of ageing plays an important role in reliability analysis. Ageing classes can be classifed according to various ageing concepts and stochastic orderings. Majority of literature on the ageing criteria consider time as countinous, with only occational references to discrete time. But disctrete lifetimes have application in different fields such as reliabilty, actuaties, biostatistics, neuroscience etc. Thus, as a discrete analoge of lapace transform ratio ordering, herewe considered *a. p. g. f*ratio ordering and derived theirproperties. Preservation property of ratio ordering under summation is established. Relation between d- NBU class and a.p.g.f. ordering is developed.

References

[1] Asha, G., Rejeesh, C.J. (2007).Models characterized by the reversed lack of memory property.*Calcutta Stat. Assoc. Bull*. **65**(233-234): 1-14.

[2] Asha, G., Elbatalb,I., Rejeesha, C. J. (2016). Further results on discrete mean past lifetime *Communications in Statistics Theory and Methods*, **45(4)**: 1081-1098.

[3]Barlow, R.E. and Marshall, A.W. (1964). Bounds for distributions with monotone hazard rate, I and II. *The Annals of Mathematical Statistics*, **35**: 1234-1257.

[4] Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinechart and Winston, Inc. NewYork.

[5] Birnbaum, Z.W., Esary, J.D. and Marshall, A.W. (1966). A stochastic characterization of wear out for components and systems. *The Annals of Mathematical Statistics*, **37**: 816-825.

[6] Bracquemond, C., Gaudoin, O. (2003). A survey on discrete lifetime distributions. *International Journal of Reliability, Quality and Safety Engineering*,**10(01):** 69-98.

[7]Bracquemond, C.,Gaudoin, O., Roy, D., &Xie, M. (2001). On some discrete notions of aging. In *System And Bayesian Reliability: Essays in Honor of Professor Richard E Barlow on His 70th Birthday* (pp. 185-197).

[8] Bryson, M.C. and Siddiqui, M.M. (1969). Some criteria for ageing. *J. Amer. Statist. Assoc.*, **64**:1472-1483.

[9] Ebrahimi, N. (1986). Classes of discrete decreasing and increasing mean residual life distributions. *IEEE Transactions on Reliability*, **35:** 403-405.

[10] Esary, J.D., Marshall, A.W. and Proschan, F. (1973). Shock Models and wear processes. *Ann. Prob.*, **1**: 627-650.

[11] Elbatal I and Ahsanullah M (2012), On Some Properties of the Discrete NBU Class Based On Generating Function Order, *Journal of Statistical Theory and Applications*, **11(3)**: 209-223

[12] Goliforushano, S., Asadi, M. (2008). On the discrete mean past time. *Metrika*. **68**:209-217.

[13] Gupta, P.L., Gupta, R.C., Tripathi, R.C. (1997). On the monotonic properties of discrete failure rate. *J. .Stat. Plann. Infer*. **65**: 255-268.

[14] Gupta, R. D., Richards, D. S. P. (1997). Invariance properties of some classical tests for exponentiality. Journal of statistical planning and inference, 63(2): 203-213.

[15] Jayamol, K.V., Jose, K.K. (2008). Stochastic ordering with respect to alternating probability generating function. *STARS, Int. Journal (Sciences)*, **2(1):** 20-28.

[16] Jayamol K V., Jose K, K (2020). Shochmodels leading to G^* class of life time distributions. *Biometrics and Biostatistics International Journal*,**9(2):** 61-66.

[17] Jiang, R. (2010). Discrete competing risk model with application to modeling bus-motor failure data. Reliab. Eng. Syst. Saf.,**95(9):** 981-988.

[18] Kemp, A.W. (2004). Classes of Discrete Lifetime Distributions. *Commu. in Stati. Theo. and Metho.*, **33**: 3069-3093.

[19] Klefsjö, B. (1982). The HNBUE and HNWUE classes of life distributions. *Naval Res. Logist. Quart.*, **29**:331-344.

[20] Klefsjö, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Prob.*, **20**: 85-616.

[21] Lai, C.D. (2013). Issues concerning constructions of discrete lifetime models. *Qual. Technol. Quant. Manag.***10(2):** 251-262.

[22] Mϋller, A.,Stoyan, D. (2002). Comparison methods for stochastic models and risks (Vol. 389). New York: Wiley.

[23] Nanda, A.K., Sengupta, D. (2005). Discrete life distributions with decreasing reversed hazard. Sankhya Ser. A. 67(1): 106-125.

[24] Proschan, F. and Hollander, M. (1984). Nonparametric concepts and methods in reliability. *Handbook of Statistics*, **4,** P.R. Krishnaiah and P.K. Sen (eds.), Amsterdam, The Netherlands, 613-655.

[25]Roy, D. and Gupta, R.P. (1999). Characterizations and model selections through reliability measures in the discrete case. *Statist. Probab.Lett.*, **43**: 197-206.

[26] Salvia, A.A., Bollinger, R.C. (1982). On discrete hazard functions. IEEE Trans. Reliab. 31: 458-459.

[27] Shaked, M. and Shanthikumar, J.G. (1994). *Stochastic Orders and Their Applications.* Academic Press, New York.

[28]Shaked, M. and Shanthikumar, J.G., and Valdez- Tottes, J.B. (1995). Discrete hazard rate functions. *Comput. Operations Res*,**22**: 391-402.

[29]Shaked, M. and Shantikumar, J.G. (2007). *Stochastic orders.* New York: Springer Science-Business Media.

[30] Shaked, M. and Wong, T. (1997). Stochastic orders based on ratios of laplace transforms. *J. Appl. Prob.*, **34**: 404-419.

[31] Stoyan, D. (1983). *Comparison models for queues and other stochastic models*. Wiley.

[32] Szekli, R. (1995). *Stochastic Ordering and Dependence in Applied Probability.* (Lecture Notes in Statistics, Vol.

97). New York: Springer-Verlag.

[33] Xekalaki, W. (1983). Hazard functions and life distributions in discrete time. textitCommun. Statist. Theory Meth., **12(21):** 2503-2509.

[34] Xie, M.,Gaudoin,D., Bracquemond, C. (2002). Redefining failure rate function for discrete distribution. *Int. J. Reliab. Qual. Saf. Eng*. **9(3):** 275-285.