A STUDY OF UNIFIED DOUBLE INTEGRAL AND LAPLACE TRANSFORMS ASSSOCIATED WITH THEM

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ABSTRACT: — Integral evaluated here involve the exponential function and the product of two general polynomials. This integral is general in nature and capable of yielding a large number of integral and double Laplace transform as their special cases. We record here two such double Laplace transform first involves the product of two general polynomials while the second involves product of Hermite polynomials and Jacobi polynomials.

INTRODUCTION: Srivastava [(3), p.1, equn. (1)] has introduced the general class of polynomials

$$S_N^M[x] = \sum_{k=0}^{[N/M]} \frac{(-N_1)_{M_K} A_{N,k} x^k}{k!}, \text{ (N=0,1,2,…..)} \quad (1.1)$$

When M is an arbitrary positive integer and coefficients $A_{N,k}(N,k \geq 0)$ are arbitrary constants real or complex . On suitably specializing the coefficients $A_{N,k}, S_N^M[x]$ yields a number of know polynomials as it special cases. These include, among others, the Jacobi polynomials, the Laguerre polynomials, the Harmite polynomials and several others [Srivastava and Singh [4,pp.158-161]

The double Laplace transform occurring herein will be defined and represented in the following manner:

$$L\{f(x_1,x_2); s_1,s_2\} = \int_0^\infty \int_0^\infty e^{-s_1x_1-s_2x_2} f(x_1,x_2)dx_1dx_2 \quad (1.2)$$

Main Integral:

$$\int_0^\infty \int_0^\infty (\lambda'_1x_1+\lambda'_2x_2)^{\sigma_1-1}(\lambda''_1x_1+\lambda''_2x_2)^{\sigma_2-1}\exp[-s_1(\lambda'_1x_1+\lambda'_2x_2)-s_2(\lambda''_1x_1+\lambda''_2x_2)]$$

$$S_{N_1}^{M_1}[C(\lambda'_1x_1+\lambda'_2x_2)^{p_1}]S_{N_2}^{M_2}[D(\lambda''_1x_1+\lambda''_2x_2)^{p_2}]dx_1dx_2$$

$$= \frac{1}{k}\sum_{k_1=0}^{[N_1/M_1]} \sum_{k_2=0}^{[N_2/M_2]} \frac{(-N_1)_{M_1k_1}(-N_2)_{M_2k_2}C^kD^k}{k_1!\ k_2!\ s_1^{(\sigma_1+p_1k_1)}s_2^{(\sigma_2+p_2k_2)}}\Gamma(\sigma_1+p_1k_1)\Gamma(\sigma_2+p_2k_2) \quad (2.1)$$

When k $=\begin{vmatrix}\lambda'_1 & \lambda''_1\\ \lambda'_2 & \lambda''_2\end{vmatrix} \neq 0$, Re $(\sigma_i) > 0$, Re $(s_i) >= 0, i = 1,2.$

Proof: we have Widder [5,p.241, eqn. (7)]

$$\int_0^\infty \int_0^\infty F(\lambda'_1x_1+\lambda'_2x_2, \lambda''_1x_1+\lambda''_2x_2)dx_1dx_2 = \frac{1}{k}\int_0^\infty \int_0^\infty F(u_1,u_2)\,du_1du_2 \quad (2.2)$$

Where k stands for the expression mentioned in (2.1)

If we take $F(\lambda'_1x_1+\lambda'_2x_2, \lambda''_1x_1+\lambda''_2x_2) = f_1(\lambda'_1x_1+\lambda'_2)f_2(\lambda''_1x_1+\lambda''_2x_2)$

Then (2.2) Transformed to

$$\int_0^\infty \int_0^\infty f_1(\lambda'_1x_1+\lambda'_2x_2,)f_2(\lambda''_1x_1+\lambda''_2x_2)dx_1dx_2 = \frac{1}{k}\int_0^\infty f_1(u_1)du_1 \int_0^\infty f_2(u_2)du_2 \quad (2.3)$$

Consider $f_1(\lambda'_1x_1+\lambda'_2x_2) = (\lambda'_1x_1+\lambda'_2x_2)^{\sigma_1-1}\exp[-s_1(\lambda'_1x_1+\lambda'_2x_2)]\,S_{N_1}^{M_1}[C(\lambda'_1x_1+\lambda'_2x_2)^{p_1}]$

$f_2(\lambda''_1x_1+\lambda''_2x_2) = (\lambda''_1x_1+\lambda''_2x_2)^{\sigma_2-1}\exp[-s_2(\lambda'_1x_1+\lambda'_2x_2)]\,S_{N_2}^{M_2}[D(\lambda'_1x_1+\lambda'_2x_2)^{p_2}]$

Then form (2.3) we get

$\int_0^\infty \int_0^\infty f_1\,(\lambda'_1x_1+\lambda'_2x_2)^{\sigma_1-1}\,(\lambda''_1x_1+\lambda''_2x_2)^{\sigma_2-1}\exp[-s_1(\lambda'_1x_1+\lambda'_2x_2)-s_2(\lambda'_1x_1+\lambda'_2x_2)]$

$S_{N_1}^{M_1}[C(\lambda'_1x_1+\lambda'_2x_2)^{p_1}]S_{N_2}^{M_2}[D(\lambda''_1x_1+\lambda''_2x_2)^{p_2}]dx_1dx_2$

$$=\frac{1}{k}\int_0^\infty u_1^{\sigma_1-1}\,e^{-s_1u_1}S_{N_1}^{M_1}\,[Cu_1{}^{p_1}]\,du_1\int_0^\infty u_2^{\sigma_2-1}\,e^{-s_2u_2}S_{N_2}^{M_2}\,[Du_2{}^{p_2}]\,du_2 \qquad (2.4)$$

On expressing the general class of polynomials occurring on the right hand side of (2.4) in terms of series with the help of (1.1) interchanging the order of integrals and summations in the result thus obtained and interchanging the u_1 and u_2 integrals . we arrived the desired result (2.1) with the help of know formula [2, p. 317, eqn. (3.381)]

SPECIAL CASE : if we take $\lambda'_1=\lambda''_2=1,\ \lambda''_1=\lambda'_2=0$, C=1,D=1, $p_1=p_2=1$ in (2.1) we get the following double Laplace transform whichinvolves the product of two general class of polynomials

$$\mathrm{L}\{x_1{}^{\sigma_1-1}x_2{}^{\sigma_2-1}S_{N_1}^{M_1}[x_1]S_{N_2}^{M_2}[x_2],s_1,s_2\}$$

$$=\sum_{k_1=0}^{[N_1/M_1]}\quad\sum_{k_2=0}^{[N_2/M_2]}\frac{(-N_1)_{M_1k_1}(-N_2)_{M_2k_2}A_{N_1,k_1}A_{N_1,k_2}}{k_1!\;k_2!\;s_1^{(\sigma_1+k_1)}s_2^{(\sigma_2+k_2)}} \qquad (2.5)$$

If we put $M_1=2, A_{N_1,K_1}=(-1)^{k_1}$ and $M_2=1\ \ A_{N_2,K_2}=\binom{N_2+\alpha}{N_2}\frac{(\alpha+\beta+N_2+1)_{k_2}}{(\alpha+1)K_2}$ in (2.5) $S_{N_1}^2[x_1]$occurring therein breaks up into $H_{N_1}\left[\frac{1}{2\sqrt{x_1}}\right]$ (known as Hermite polynomials) [4, P. 158 eqn. (1.4)] and the polynomial $S_{N_2}^2[x_2]$ reduces to $P_{N_2}^{(\alpha,\beta)}[1-2x_2]$ (Jacobi polynomials) [4, p. 159. Eqn. (1.6)] and we get the following Laplace Transform involving the product of the Hermite polynomials and the Jacobi polynomials.

$$\mathrm{L}\left\{x_1^{\alpha,-1}x_2^{\sigma_2-1}H_{N_1}\left[\frac{1}{2\sqrt{x_1}}\right]P_{N_2}^{(\alpha,\beta)}[1-2x_2];s_1,s_2\right\}$$

$$=\sum_{k_1=0}^{[N_1/2]}\sum_{k_2=0}^{[N_2]}\frac{(-N_1)_{2k_1}(-N_2)_{k_2}(-1)^{k_1}}{k_1!k_2!}\binom{N_2+\alpha}{N_2}\frac{(\alpha+\beta+N_2+1)_{k_2}}{(\alpha+1)\boldsymbol{k_2}}s_1{}^{-(\sigma_1+k_1)}s_2{}^{-(\sigma_2+k_2)} \qquad (2.6)$$

If we put in (2.1) $\sigma_1=\sigma_2=M_1=M_2=s_1=s_2=1\ p_1=p_2=C=D=1$ and replace $A_{N_1,k_1}\ by\ \binom{N_1+\alpha_1}{N_1}$ and $A_{N_2,k_2}\ by\ \binom{N_2+\alpha_2}{N_2}$ respectively , then $S_{N_1}^{M_1},S_{N_2}^{M_2}$ occuring therein reduce to laguerre polynomials Srivastava and Singh [4, p.159, eqn (1.8)] we arrive at a known result Dhawan [1, p. 417, eqn (2.2)] after a little simplification.

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