ON COMPOSITION OF MULTIDIMENSIONAL INTEGRAL OPERATORS INVOLVING GENERAL POLYNOMIALS AND A MULTIVARIABLE H-FUNCTION

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ABSTRACT : In the present paper we obtain interesting composition formulae of a class of multidimensiona1 fractional integral operators involving the product of the generalized polynomials $S_{N_1 \ldots N_s}^{M_1 \ldots N_s}[x_1, \ldots, x_s]$ and the multivariable H-function.On account of the most general nature of the functions used here as kernels , the main results of our paper are unified in nature and capable of yielding a very large number of corresponding results (new and known) involving simpler special function and polynomials (of one or more variables) as special cases of our formulae.

INTRODUCTION

In this paper we shall establish composition formulae for the multidimensional fractional integral operator defined by means of the following equations :

$$I_s[f(t_1, \ldots, t_s)] = I_{X_s\ \ : z_s\ ;E_s}^{\rho_s,\sigma_s:\, h_s\ ;\ v_s}[f(t_1, \ldots, t_s); x_1, \ldots, x_s]$$

$$= \prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j} \int_0^{x_1} \cdots \int_0^{x_s} \left(\prod_{j=1}^{s} t_j^{\rho_j} (x_j - t_j)^{\sigma_j - 1} \right)$$

$$S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\left[z_1 \left(1 - \frac{t_1}{x_1}\right)^{h_1}, \cdots, z_s \left(1 - \frac{t_s}{x_s}\right)^{h_s} \right]$$

$$H\begin{bmatrix} E_1\left(1-\frac{t_1}{x_1}\right)^{v_1} \\ \vdots \\ E_s\left(1-\frac{t_s}{x_s}\right)^{v_s} \end{bmatrix} f(t_1, \ldots, t_s) dt_1 \cdots dt_s \tag{1.1}$$

The multivariable H - function occurring in the equations (1.1) is defined and represented *as* follows [2, pp. 251 - 252, eqns . (c.2) - (c .3)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix} = H_{p,q:p_1,q_1,\ldots,p_s,q_s}^{o,n:m_1,n_1,\ldots,m_s,n_s}\begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix}\begin{matrix} (a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(s)})_{1,p}: \\ (b_j, \beta_j^{(1)}, \ldots, \beta_j^{(s)})_{1,q}: \end{matrix}$$

$$\left.\begin{matrix} (c_j^{(1)}, \gamma_j^{(1)})_{1,p_1}; \ldots; (c_j^{(s)}, \gamma_j^{(s)})_{1,p_s} \\ (d_j^{(1)}, \delta_j^{(1)})_{1,q_1}; \ldots; (d_j^{(s)}, \delta_j^{(s)})_{1,q_s} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^{\delta}} \int_{L_1} \cdots \int_{L_s} \phi_1(\xi_1) \cdots \phi_s(\xi_s) \psi(\xi_1, \cdots, \xi_s)$$

$$z_1^{\xi_1} \cdots z_s^{\xi_s} d\xi_1 \cdots d\xi_s \tag{1.2}$$

The convergence conditions of integral given by (1.2) and other details of the multivariable H-function can be seen in the book referred above.

During our study we shall also need a second multivariable H-function. We shall denote it by the symbol

$$H^*\begin{bmatrix} z'_1 \\ \vdots \\ z'_s \end{bmatrix}$$

We shall represent this function in the following manner :

$$H^*\begin{bmatrix} z'_1 \\ \vdots \\ z'_s \end{bmatrix} = H^{o,n':m'_1,n'_1,\dots,m'_s n'_s}_{p',q':p'_1,q'_1\dots,p'_s,q'_s}\begin{bmatrix} z'_1 \\ \vdots \\ z'_s \end{bmatrix}\begin{matrix}(a'_j,\alpha'^{(1)}_j,\dots,\alpha'^{(s)}_j)_{1,p'}: \\ (b'_j,\beta'^{(1)}_j,\dots,\beta'^{(s)}_j)_{1,q'}:\end{matrix}$$

$$\left.\begin{matrix}(c'^{(1)}_j,\gamma'^{(1)}_j)_{1,p'_1};\dots:(c'^{(s)}_j,\gamma'^{(s)}_j)_{1,p'_s} \\ (d'^{(1)}_j,\delta'^{(1)}_j)_{1,q'_1};\dots:(d'^{(s)}_j,\delta'^{(s)}_j)_{1,q'_s}\end{matrix}\right] \tag{1.3}$$

Also, the general polynomials $S^{M_1\dots N_s}_{N_1\dots N_s}[x_1,\dots,x_s]$occurring in the definitions of our operators of study stands for the multi-variable polynomials given by Srivastava [4, p . 185 , eqn. (7)) defined and represented in the following slightly modified form

$$S^{M_1\dots N_s}_{N_1\dots N_s}[x_1,\dots,x_s] = \sum_{k_1=0}^{|N_1/M_1|}\cdots\sum_{k_s=0}^{|N_s/N_s]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_s)_{M_sk_s}}{k_s!} A[N_1,k_1;\dots,N_s,k_s]x_1^{k_1}\cdots x_s^{k_s}, \tag{1.4}$$

where N i = 0, 1, 2, ... ; $M_i \neq 0$[i = l , ... , s] M_i is an arbitrary positive integer. The coefficients $A[N_i, k_i ; \dots ; N_s, k_s]$ being arbitrary constants, real or complex.
If we take $s = 1$ in the equation (1.4) and denote A $[N$, k] thus obtained by $A_{N,k}$, we arrive at the well know general class of polynomials introduced ,by Srivastava[3, p.158, eqn. (1.1)].

MAIN RESULT :

$$I_s[f(t_1,\dots,t_s)] = I^{\rho_s,\sigma_s:h_s;v_s}_{X_s\ \ :z_s;E_s} I^{\rho'_s,\sigma'_s:h'_s;v'_s}_{y'_s\ \ :z'_s;E'_s}[f(t_1,\dots,t_s)]$$

$$= \prod_{j=1}^s x^{-\rho'_j-1}\int_0^{x_1}\cdots\int_0^{x_s}\prod_{j=1}^s t_j^{\rho'_j} f(t_1,\dots,t_s) G\left(\frac{t_1}{x_1},\dots,\frac{t_s}{x_s}\right) dt_1\dots dt_s \tag{2.1}$$

Where

$$G(t_1,\dots t_s) = \sum_{k'_1=0}^{[N'_1/M'_1]}\cdots\sum_{k'_s=0}^{|N'_s/M'_s|}\sum_{k_1=0}^{[N_1/M_1|}\cdots\sum_{k_s=0}^{[N_s/M_s]}\prod_{j=1}^{s}\left(\frac{(-N'_j)_{M'_jk'_j}}{k'_j!}z_j^{k'_j}\right)A[N'_1,k'_1;\dots,N'_s,k'_s]$$
$$\prod_{j=1}^s\left(\frac{(-N_j)_{M_jk_j}}{k_j!}z_j^{k_j}\right)A[N_1,k_1;\dots,N_s,k_s]\left(\prod_{j=1}^{s}(1-t_j)^{\rho_j+\sigma'_j+h_jk_j+h'_jk'_j-1}\right)$$

$$H^{o,n'+n+2s\ :m'_1,n'_1+1;\dots;m'_s,n'_s+1;m_1,n_1;\dots;m_s,n_s:1,o;\dots;1,o}_{p'+p+2s,q'+q+s:p'_1+1,q'_1+1;\dots;p'_s+1,q'_s+1;p_1,q_1;\dots;p_s,q_s:o,1;\dots;o,1}\left[\begin{matrix}E'_1(1-t_1)^{v'_1} \\ \vdots \\ E'_s(1-t_s)^{v'_s} \\ E_1(1-t_1)^{v_1} \\ \vdots \\ E_s(1-t_s)^{v_s} \\ -(1-t_1) \\ \vdots \\ -(1-t_s)\end{matrix}\ \middle|\ \begin{matrix}A & : & C \\ \\ B & : & D\end{matrix}\right] \tag{2.2}$$

Also here

$$A = \left(a_j,\frac{o,\dots,o}{s},\alpha_j^{(1)},\dots,\alpha_j^{(s)},\frac{o,\dots,o}{s}\right)_{1,n}(1+\rho_1-\rho'_1-\sigma'_1-h'_1k'_1;$$

$$v'_1,\frac{o,\dots,o}{2s-1},1,\frac{o,\dots,o}{s-1}),\dots,(1+\rho_s-\rho'_s-\sigma'_s-h'_sk'_s;\frac{o,\dots,o}{s-1},v'_s$$

$$\frac{o,\dots,o}{2s-1},1),(1-\sigma_1-h_1k_1;\frac{o,\dots,o}{s},v_1,\frac{o,\dots,o}{s-1},1,\frac{o,\dots,o}{s-1}),\dots \tag{2.3}$$

$$(1-\sigma_s-h_sk_s;\frac{o,\dots,o}{2s-1},v_s,\frac{o,\dots,o}{s-1},1)(a'_j,\alpha_j'^{(1)},\dots,\alpha_j'^{(s)},\frac{o,\dots,o}{2s})_{1,p'}$$

$$(a_j;\frac{o_1\dots,o}{s},\alpha_j^{(1)},\dots,\alpha_j^{(s)},\frac{o,\dots,o}{s})_{n+1,p}$$

$$B=(b'_j,\beta_j'^{(1)},\dots,\beta_j'^{(s)},\frac{o,\dots,o}{2s})_{1,q'}(b_j;\frac{o,\dots,o}{s},\beta_j^{(1)},\dots,\beta_j^{(s)},\frac{o,\dots,o}{s})_{1,q}$$

$$\left(1-\sigma_1-\sigma'_1-h_1k_1-h'_1k'_1;v'_1,\frac{o,\dots,o}{s-1},v_1,\frac{o,\dots,o}{s-1},1,\frac{o,\dots,o}{s-1}\right),$$

$$\cdots,(1-\sigma_s-\sigma'_s-h_sk_s-h'_sk'_s;\frac{o,\dots,o}{s-1},v'_s,\frac{o,\dots,o}{s-1},v_s,\frac{o\dots o}{s-1},1) \tag{2.4}$$

$$C=(1-\sigma'_1-h'_1k'_1,v'_1),\left(c_j^{(1)},\gamma_j'^{(1)}\right)_{1,p'_1};\dots;(1-\sigma'_s-h'_sk'_s,v'_s),$$

$$(c_j'^{(s)},\gamma_j'^{(s)})_{1,p'_s},(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\dots\left(c_j^{(s)},\gamma_j^{(s)}\right)_{1,p_s};-;\dots;-$$

$$D=\left(d_j'^{(1)},\delta_j'^{(1)}\right)_{1,q'_1}(1+\rho_1-\rho'_1-\sigma'_1-h'_1k'_1,v'_1),\dots,\left(d_j^{(s)},\delta_j'^{(s)}\right)_{1,q'_s}$$

$$(1+\rho_s-\rho'_s-\sigma'_s-h'_sk'_s,v'_s),\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1},\dots,\left(d_j^{(s)},\delta_j^{(s)}\right)_{1,q_s},\frac{(o,1);\dots;(o,1)}{s} \tag{2.5}$$

Again, $\frac{o,\dots o}{s}$ occurring on the right hand side of the equation (2.3) and (2.4) would means S zeros, and so on.
Provided that Re $(p_j+u_j+1)>0$,

$$\min \text{Re}\,[\rho_j+v_j(\frac{d_k^{(j)}}{\delta_k^{(j)}})]>0,\min \text{Re}\,[\rho_j+v_j(\frac{d_k^{(j)}}{\delta_k^{(j)}})]>0$$

$$(k=1,\dots,m_j,k'=1,\dots,m'_j;j=1,\dots,s).$$

Proof: To prove the result 1, we first express the *I*-operators involved in its left hand side in the integral form with the help of equation (1.1). Next, we interchange the order of t_j- and y_j- integral (which is permissible under the conditions stated by a known multidimensional extension of Fubini theorem), we obtain

$$I_{X_s}^{\rho_s,\sigma_s:\,h_s\,;\,v_s}{}_{:\,z_s\,;E_s}I_{y'_s}^{\rho'_s,\sigma'_s:\,h'_s\,;\,v'_s}{}_{:\,z'_s\,;E'_s}[f(t_1,\dots,t_s)]=\prod_{j=1}^{s}x_j^{-\rho_j-\sigma_j}\int_0^{x_1}\cdots\int_0^{x_s}\prod_{j=1}^{s}t_j^{\rho'_j}f(t_1,\dots,t_s)\Delta dt_1\dots dt_s$$

Where (2.6)

$$\Delta=\int_{t_1}^{x_1}\cdots\int_{t_s}^{x_s}\prod_{j=1}^{s}\left[y_j^{\rho_j-\rho'_j-\sigma'_j}(x_j-y_j)^{\sigma_j-1}(y_j-t_j)^{\sigma'_j-1}\right]$$

$$S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\left[z_1\left(1-\frac{y_1}{x_1}\right)^{h_1},\cdots,z_s\left(1-\frac{y_s}{x_s}\right)^{h_s}\right]H\begin{bmatrix}E_1\left(1-\frac{y_1}{x_1}\right)^{v_1}\\ \vdots\\ E_s\left(1-\frac{y_s}{x_s}\right)^{v_s}\end{bmatrix}$$

$$S_{N'_1,\dots,N'_s}^{M'_1,\dots,M'_s}\left[z'_1\left(1-\frac{t_1}{y_1}\right)^{h'_1},\cdots,z'_s\left(1-\frac{t_s}{y_s}\right)^{h'_s}\right]H^*\begin{bmatrix}E'_1\left(1-\frac{t_1}{y_1}\right)^{v'_1}\\ \vdots\\ E'_s\left(1-\frac{t_s}{y_s}\right)^{u'_s}\end{bmatrix} \tag{2.7}$$

$$dy_1\cdots dy_s$$

To evaluate Δ.we first express both the generalized polynomials in terms of series with the help of equation (1.4). Now we change the order of these series and y_j- integrals $(j=1,\dots,s)$. Next, we express both the mutivariable H-function in terms of Mellin-Barnes type contour integrals. Now we interchange the order of $\xi_j, -\xi_j, -$and $y_j -$ integrals $(j = 1, \dots, s)$ (which is permissible under the conditions stated), evaluate the $y_j -$ integrals by setting

$$u_j = \frac{x_j - y_j}{x_j - t_j} (j = 1, \dots, s)$$

And using the known formula [5, p. 287, eq(8)]. On reinterpreting the result thus obtained in terms of the multivariable H-function, we get the value of Δ. Now on substituting the value of Δ. thus obtained in the equation (2.6), we arrived at the result after a little simplification.

3. Special cases

If we take $s = 1, n = p = q = n' = p' = q' = 0$ and $v_1 = v_1' = 0$ in (2.1), and reduce the H-functions

$$H_{p_1,q_1}^{m_1,n_1}, H_{p_1',q_1'}^{m_1',n_1'}$$

thus obtained to exponential function, let them tend to zero and reduce the polynomials $S_{N_1}^{M_1}$and $S_{N'_1}^{M'_1}$to unity by putting $N_1 = N'_1 = 0$, we get the corresponding expressions which are in essence the same as those given by Erdélyi [1,p. 166, eq. (6.1)]

Also, if we take in (2.1), $s = 1, n = p = q = n' = p' = q' = 0$, $v_1 = v_1' = 1$, and reduce the H-functions thus obtained to generalized hyper geometric functions, we get the expressions which are the same in essence as obtained by Goyal et al. [6, p. 404-405, eq. (2.)]

References:

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