MATHEMATICAL MODELLING OF ELECTRICAL SYSTEMS

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Abstract

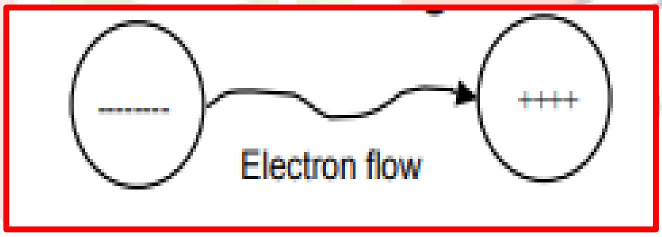
In case of system Mathematical model plays an important role to give response. This paper explains different kinds of system such as electrical and mechanical. In Accordance of it examples of Mechanical, Electrical system are represented by mathematical model; in different types of Mathematical model i.e. Mechanical System by Differential Equation Model, Electrical system by State-Space Model and Transfer Function Model. To describe the behaviour of electrical circuits involving resistors, capacitors, and inductors using mathematical equations. This model using analysis, design and Mathematical modelling of electromechanical systems.

Keywords: Mathematical modelling, Electrical, Mechanical systems and their behaviour.

Basics Terminology:

We Know that the basics terminology are as follows.

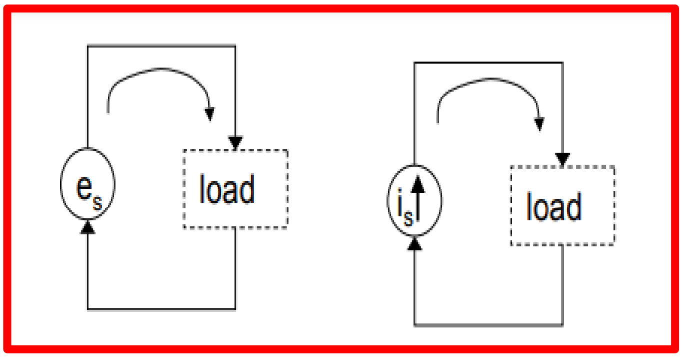
•Voltage: It is the electromotive force needed to produce the flow of electrons (analogous to pressure). The unit of voltage is volt (V).



•Charge: It is proportional to the excess of electrons (negative charge) over protons (positive charge) in a matter. The unit of charge is coulomb. 1 coulomb = charge of 6.25×1018 electrons
•Current: It is the rate of flow of electrical charge (coloumb/sec=amp):

$$i = \frac{dq}{dt} \Leftrightarrow q = \int i\,dt$$

Power Sources:
There are two types of electrical power sources. Which are as follows.
•Voltage source delivers a specified voltage regardless of the current drawn by the load
• Current Source delivers a specified current regardless of the voltage drop across the load



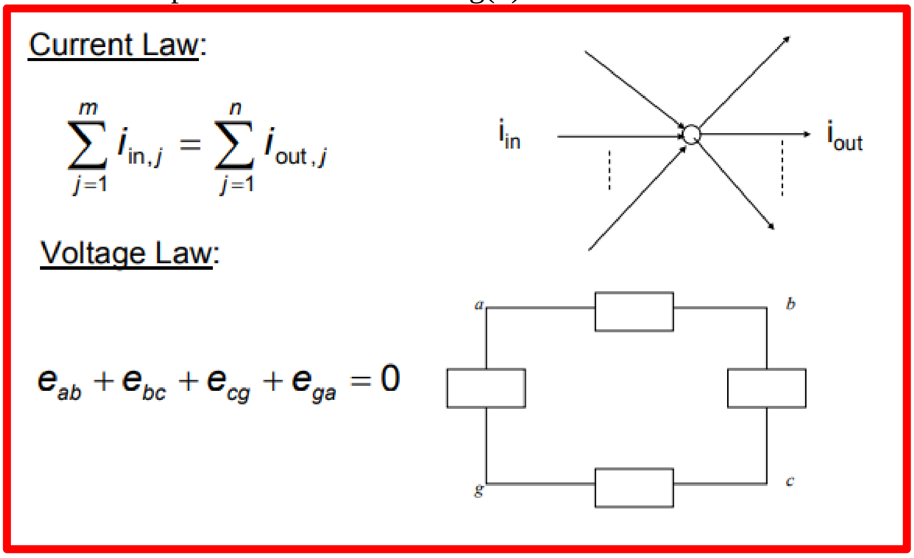
Fig(1) Voltage and Current Source

Basic Electrical Elements: The basic electrical elements are shown in table(1)

Table(1)

Element	Symbol	Constitutive Relationship
Resistor	a, R, b, i_R, e_R	Voltage Drop ∝Current $e_R = e_a - e_b = Ri_R$
Capacitor	a, C, b, i_c, e_C	Voltage Drop ∝Charge e_C=q/C, C: Capacitance (F) $\frac{de_C}{dt} = \frac{1}{C} i_C \Leftrightarrow e_C = \frac{1}{C}\int i_C dt$
Inductor	a, L, b, i_L, e_L	Voltage Drop ∝d(Current)/dt e_L=Ldi_L/dt, L: Inductance (H) $\frac{di_L}{dt} = \frac{1}{L} e_L \Leftrightarrow i_L = \frac{1}{L}\int e_L dt$

Kirchhoff's Current & Voltage Laws: The Kirchhoff 's law are two types one is Kirchhoff 's Current Law(KCL) and Kirchhoff 's Voltage Law(KVL).
According to KCL the algebraic sum of all currents at any node should be equal to zero.
And According to KVL the algebraic sum of Voltage of each element in a closed loop should be equal to the voltage source in that closed loop. Which is shown in fig(2).



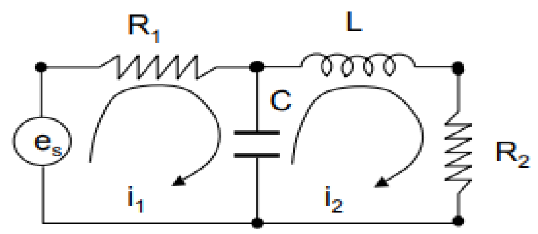
Fig(2) KVL and KCL

Mathematical Modelling: For mathematical modelling generally the following,
There are 3 commonly used methods are used. Which are as follows.
Loop Method „
Node Method „
Complex Impedances

Loop Method:
1.First of all Label each node and assign cyclic current to each loop.
2.Now Write the Kirchhoff Voltage law for each loop. Express voltage across each element in terms of the cyclic currents assigned in step.
3.Now Assemble the equations into a set of differential/integral equations with an equal number equations and unknown variables.
4.If desired eliminate all the integrals using the relationship $q=\int i\,dt$ to obtain a system of differential equations.

Node Method:
1.First of all Label each node.
2.Now Apply the Kirchhoff-current law at each node.
3.Now Express all currents in terms of nodal or elemental voltages.
4.After that Assemble the equations into a set of ordinary differential equations with an equal number equations and unknown variables. Eliminate integrals using $i_L=\int v_L dt$ for all the inductors.

Loop Method Example:



Fig(3) Example for Loop Method

From Fig (3) The loop equations are given by
Loop equations: $R_1 i\ 1+(1/C)\int(i_1-i_2)dt=e_s$
$Ldi_2/dt+R_2 i_2+(1/C)\int(i_2-i_1)\ dt=0$
How to get rid of integrals? Try using charges $q_1= \int i_1 dt$ and $q_2= \int i_2 dt$ instead of i_1 and i_2: $R_1 dq_1/dt+(1/C)(q_1-q_2) = e_s$
$Ld^2q_2/dt^2+R_2 dq_2/dt +(1/C)(q_2-q_1) = 0$

Method of Complex Impedances:

- First of all Transforms the circuit directly into the Laplace domain by using each elements Impedances .
- Results in input-output transfer function.
- This method is described in the text pages and have been.
- Illustrated in classroom.

Mixed Systems:
- The Most systems in mechatronics are of the mixed type, e.g., electromechanical, hydro mechanical, etc
- And Each subsystem within a mixed system can be modelled as single discipline system first
- Power transformation among various subsystems are used to integrate them into the entire system
- Overall mathematical model may be assembled into a system of equations, or a transfer function

D.C. Motor Illustrating D.C. Motor:

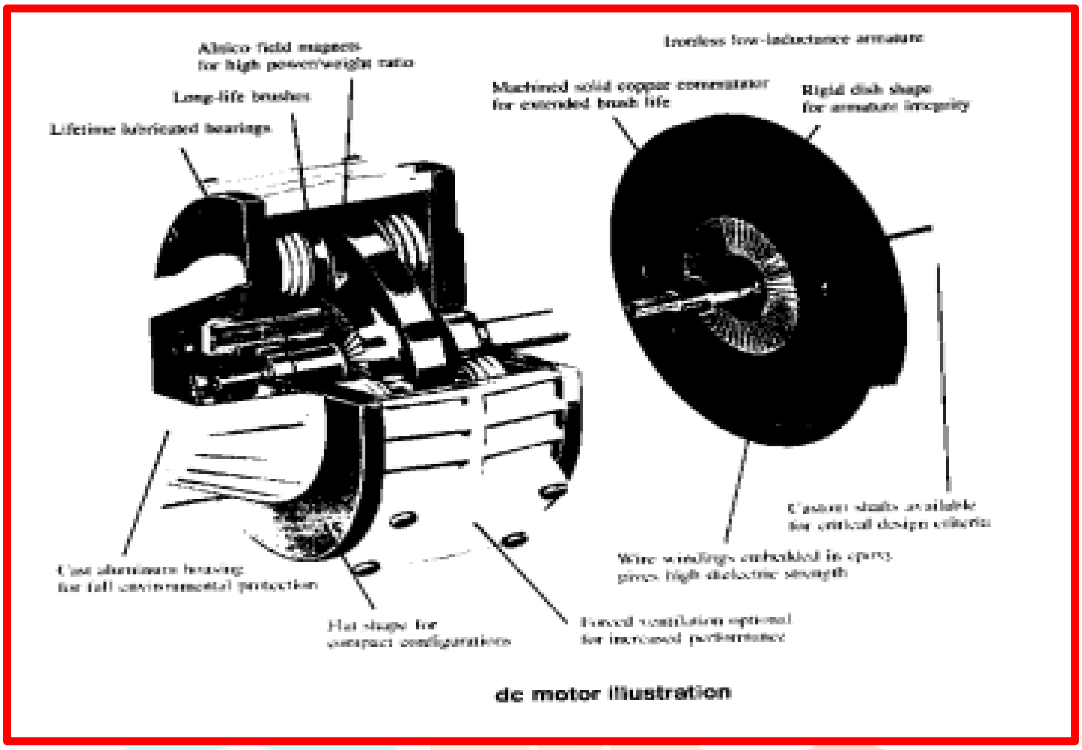


Fig.(4) Illustrating DC Motor

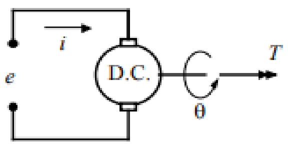
D.C. Motor Model:

D.C. Motor converts electrical power to mechanical power. It can be used as a velocity or position actuator.

•Motor torque is a function of armature current

•Motor angular velocity is proportional to its (back-emf) voltage.

Schematic:



$T = K_t \cdot i$ K_t: Torque Constant

$e = K_b \cdot \omega$ K_b: Velocity Constant

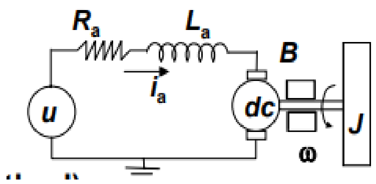
Ideal Motor: $K_t = K_b \Rightarrow$ Input Power=ei=Output Power=$T\omega$

Electro-Mechanical Example:

Input: Voltage u

Output: Angular Velocity ω

Electrical Subsystem (Loop Method):



$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b,\ e_b = back\text{-}emf\ voltage$$

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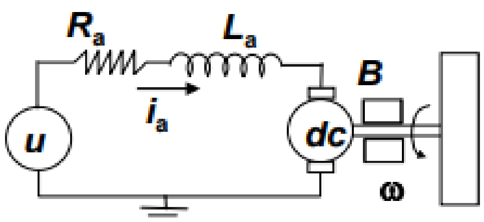
Output: Angular Velocity ω
Electrical Subsystem (Loop Method):

$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b,\ e_b = back\text{-}emf\ voltage$$

Mechanical Subsystem
$T_{motor} = J\dot{\omega} + B\omega$

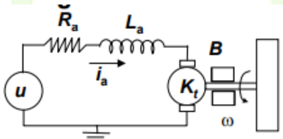


Power Transformation:
Torque-Current: $T_{motor} = K_t\, i_a$
Voltage-Speed: $e_b = K_b\, \omega$
Where K_t: Torque Constant, K_b: Velocity Constant
For an ideal motor
Combing previous equations results in the following mathematical model:

$$L_a \frac{di_a}{dt} + R_a i_a + K_b \omega = u$$
$$J\dot{\omega} + B\omega - K_t i_a = 0$$

Transfer Function of Electromechanical Example:
Taking Laplace transform of the system's differential equations with zero initial conditions gives:

$$\begin{cases}(L_a s + R_a) I_a(s) + K_b \Omega(s) = U(s) \\ (Js + B)\Omega(s) - K_t I_a(s) = 0\end{cases}$$



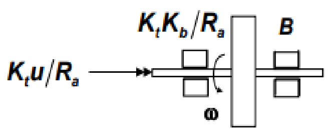
Eliminating I_a yields the input-output transfer function

$$\frac{\Omega(s)}{U(s)} = \frac{K_t}{L_a J s^2 + (JR_a + BL_a) + BR_a + K_t K_b}$$

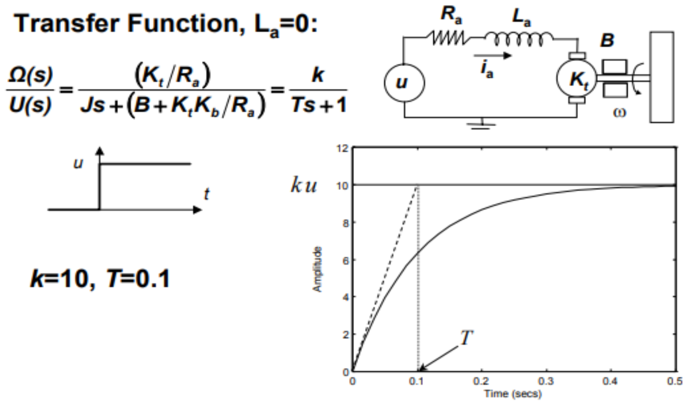
Reduced Order Model:
Assuming small inductance, La ≈0

$$\frac{\Omega(s)}{U(s)} = \frac{(K_t/R_a)}{Js + (B + K_t K_b/R_a)}$$

which is equivalent to



The D.C. motor provides an *input torque* and an additional *damping* effect known as back-emf damping



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•Power transformation among various subsystems are used to integrate them into the entire system.

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TRANSFER FUNCTION:

It has been shown already that the input and output of a linear system in general, is related by a linear or a set of linear differential equations. Such relationships are capable of completely describing the system behaviour in the presence of a particular input excitation and known initial conditions.

differential equation of Eq.(A) is seldom used in its original form for the analysis and design of control systems. To obtain the transfer function of the linear system that is represented by Eq.(A), we simply take the Laplace transform on both sides of the equation, and assume zero initial conditions. The result is

$$(s^n + a_n s^{n-1} + \dots + a_2 s + a_1)C(s) = (b_{m+1}s^m + b_m s^{m-1} + \dots + b_2 s + b_1)R(s) \qquad \textbf{(A)}$$

The transfer function between r(t) and c(t) is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_{m+1}s^m + b_m s^{m-1} + \dots + b_2 s + b_1}{s^n + a_n s^{n-1} + \dots + a_2 s + a_1} \qquad \textbf{(B)}$$

We can summarize the properties of the transfer function as follows:

1.Transfer function is defined only for a linear time-invariant system. It is meaningless for nonlinear systems.

2.The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response. Alternately, the transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input.

3.When defining the transfer function, all initial conditions of the system are set to zero.

4.The transfer function is independent of the input of the system.

5.Transfer function is expressed only as a function of the complex variable s. It is not a function of the real variable, time, or any other variable that is used as the independent variable.

Transfer Function (Multivariable Systems) :

The definition of transfer function is easily extended to a system with a multiple number of inputs and outputs. A system of this type is often referred to as the multivariable system. In a multivariable system, a differential equation of the form of Eq. (**A**) may be used to describe the relationship between a pair of input and output variables. When dealing with the relationship between one input and one output, it is assumed that all other inputs are set to zero. Since the principle of superposition is valid for linear systems, the total effect on any output variable due to all the inputs acting simultaneously is obtained by adding up the outputs due to each input acting alone. A number of examples is appropriate to illustrate the concept of transfer function.

Conclusion: In Order to understand the behaviour of systems, Mathematical Models are needed. These are simplified representations of certain aspects of real system. Such a model is created using equations to describe the relationship between input and output of system and can then be used to enable prediction to be made of the behaviour of a system under specific condition.

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